Revisiting the “Decentralization Theorem”—On the role of externalities

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Abstract

The “Decentralization Theorem” [Oates, W.E., 1972. Fiscal Federalism. Harcourt Brace Jovanovich, New York] is central to the discussion of fiscal federalism. We revisit the role of consumption spillovers in evaluating the merits of (de)centralization. Unlike the general prediction, a higher degree of spillovers may reduce the difference in utility of centralization and decentralization. The non-monotonicity result relates to the difference in expenditures on public consumption. Provided decentralized choices yield higher levels of public expenditure, a rise in the amount of spillovers allows residents to enjoy larger gains in public consumption (and thereby utility) under decentralization relative to centralization.

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1. Introduction

The question of whether fiscal responsibilities should be assigned to a (de)centralized authority has long been debated in public economics. The discussion refers to Oates’ Decentralization Theorem (Oates, 1972) stating that in the absence of cost savings from centralization and interjurisdictional externalities, fiscal responsibilities should be decentralized. This argument implicitly assumes that the center is unresponsive to preference heterogeneity and thereby is only able to implement uniform policies. More specifically, “[...] individual local governments are presumably much closer to the people [...] they posses knowledge of both local preferences and cost conditions that a central agency is unlikely to have” (Oates, 1999, p. 1123).1 If the geographical scope of a jurisdiction falls short of the spatial pattern of spending benefits, the optimal assignment of policy tasks is deduced by trading off the welfare costs of policy uniformity against the welfare gains from internalizing spillovers in policy-making.2

Consider a country consisting of two regions which differ in their preferences for local public goods, which

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1 The failure to adapt to taste differences is central to recent analyses of fiscal federalism—see e.g. Alesina and Spolaore (1997), Bolton and Roland (1997), Tabellini (2003), Brueckner (2004), Alesina et al. (2005), and Brueckner (2006).

2 The posited trade-off is the essence of much of the discussion not only related to fiscal unions, but also to monetary unions and free-trade areas; see e.g. Alesina and Barro (2002).
exhibit regional spillovers. In this setting, fiscal decentralization allows for a better matching of public good provision to local tastes, whereas under centralization uniform provision ignores local taste heterogeneity, but internalizes spillovers. The central question to be examined in this paper is how the difference in the utility of centralization and decentralization changes with respect to the level of consumption spillovers. Using quasi-linear, iso-elastic preferences, the welfare difference turns out to be non-monotone in the strength of spillovers. A larger amount of spillovers may reduce the welfare differential between centralization and decentralization. The rationale for this result is that decentralization may yield higher expenditures on public goods than centralization. In fact, more resources are spent on public goods under decentralized decision-making when spillovers are not too large and the demand for public consumption is sufficiently elastic. In this case, a rise in spillovers gives residents higher utility gains when fiscal authority is decentralized, due to the fact that the higher decentral spending allows residents to enjoy a larger increase in public consumption (and thereby utility) in response to a hike in the level of spillovers. The finding may be unexpected given the virtue of centralization to internalize spillovers.

We further show that a non-monotonicity of the welfare difference only arises when decentralization yields higher welfare. As such, an increase in the amount of spillovers reinforces the welfare-superiority of decentralized decision-making but, more importantly, will not justify a reassignment of fiscal authority from the central level to the regional level. Hence, the paper’s finding does not invalidate the bottom line of the “Decentralization Theorem” that centralization (decentralization) yields higher welfare when spillovers are sufficiently high (low).

To the best of our knowledge, the result has not been mentioned in previous analyses of fiscal federalism, which resort to a uniformity-externality trade-off. The paper complements earlier political-economy research on the merits of (de)centralization (Lockwood, 2002, and Besley and Coate, 2003). Therein, the equilibrium policy entails regionally differentiated public good bundles. The welfare trade-off follows differently from the political-economy deficiencies of centralized systems weighed up against the failure of internalization in decentralized systems. Relative welfare may not vary monotonically with the strength of consumption spillovers. The finding reflects inefficiencies inherent either to the formation of minimum winning coalitions or to the strategic delegation of politicians to a central legislature. Both types of political deficiencies are absent in our model. Instead, we resort to an archetypal model of fiscal federalism hypothesizing benevolent governments; a setting which is most susceptible to predicting a monotone uniformity-externality trade-off.

The remainder of the paper is organized as follows. Section 2 introduces the model. The welfare analysis of (de)centralization is provided in Section 3. Section 4 concludes.

2. The model

Private Sector Consider 2 regions each being inhabited by a representative household whose preferences are defined over private and public consumption and are quasi-linear in private consumption, $u_i(c_i, G_i) = c_i + \theta_i v(G_i)$ where $\theta_i > 0$, $v(G_i)$ is continuously differentiable and satisfies $v'(G_i) > 0$, $v''(G_i) < 0$, and $\lim_{G_i \to 0} v'(G_i) = \infty$. Private consumption $c_i$ equals the endowment $I_i$ minus taxes levied by the government $t_i$, $c_i = I_i - t_i$. Public consumption in region $i$ is $G_i = g_i + \alpha g_i$, $i \neq j$. Region $i$ benefits from resources spent on public consumption in the neighbor state at a rate $\alpha \in [0, 1]$.

Public Sector There are two types of policy regimes. With a central legislature we assume that public good are uniformly provided as conjectured by Oates (1972). Rather than imposing uniform policy choices, we could

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3 In general, two types of welfare comparisons can be distinguished. The first type looks at the welfare difference as a function of spillovers irrespectively of the sign of the difference. The second type confines its attention to spillover values at which the sign of the welfare differential changes. The paper primarily deals with the former.

4 Interestingly, the contributions point to a normative interpretation of Oates’ uniformity assumption. With a centralized legislature, an exogenously imposed policy uniformity requirement potentially enhances welfare (as policy uniformity reduces the incentives for pork-barrel spending).

5 We refrain from a pure public good ($\alpha = 1$), not because it is less important, but because policy uniformity would be inherently related to the nature of the public good, rather than a deficiency of a central legislature.

6 A perfect separation of preference types may follow from Tiebout-type sorting (Tiebout, 1956). The analysis extends to heterogeneous populations which imperfectly sort across regions; most straightforwardly when the public good $g_i$ is pure from region $i$’s perspective. In this case $\theta_i$ captures the average preference type in region $i$. Please cite this article in press as: M. Koethenbuerger, Revisiting the “Decentralization Theorem”—On the role of externalities, Journal of Urban Economics (2008), doi:10.1016/j.jue.2007.10.001
alternatively treat policy uniformity as an endogenous, equilibrium outcome along the lines suggested by Klibanoff and Poitevin (1999).7

The central government’s choice of public expenditure $g$ follows from

$$\max_{g} I^{h} + I^{l} - 2t + (\theta^{h} + \theta^{l}) v((1 + \alpha)g) $$

s.t. $t = g$.

The first-order condition is

$$1 + \alpha)\bar{\theta}v'(1 + \alpha)g) = 1, \tag{1}$$

where $\bar{\theta} := \frac{\theta^{h} + \theta^{l}}{2}$. (1) implicitly defines the optimal level of public expenditure, denoted by $\tilde{g}$, as a continuous function of $\alpha, \theta^{h}$ and $\theta^{l}$. The policy choice reflects the expenditure spillover $\alpha$. Due to the uniform provision of public goods, it is however only optimal for a hypothetical region endowed with average preferences $\bar{\theta}$.

With decentralization, each region independently determines its most preferred level of public expenditure. Taking the policy of the neighbor state $g^{j}$ as given, the government in region $i$ solves

$$\max_{g^{i}} I^{i} - t^{i} + \theta^{i} v(g^{i} + \alpha g^{j})$$

s.t. $g^{i} = t^{i}, i \neq j$.

In public good contribution games, agents may optimally decide not to contribute to the public good; even when $v(G^{i})$ goes to infinity as $G^{i}$ becomes small (Bergstrom et al., 1986). We only consider equilibria in which $\{g^{i} > 0\}_{i=1,2}$. In such equilibria, both contribution margins adjust in response to a rise in the amount of spillovers, and the induced change in the efficiency cost of decentralization (due to free-riding) is presumably most pronounced. Finding that the difference in utility of centralization and decentralization is non-monotone in the strength spillovers is possibly most unexpected in this setting. At an interior solution, the first-order condition

$$\theta^{i} v'(g^{i} + \alpha g^{j}) = 1, \quad i \neq j, \tag{2}$$

yields the optimal best-response as a continuous function of $\alpha, \theta^{i}$ and $g^{i}$, i.e. $\tilde{g}^{i} = r^{i}\left(\tilde{g}^{j}, \alpha, \theta^{i}\right)$. Mutually consistency of responses requires that $\tilde{g}^{i} = r^{i}\left(\tilde{g}^{j}, \alpha, \theta^{i}\right)$ and $\tilde{g}^{j} = r^{j}\left(\tilde{g}^{i}, \alpha, \theta^{j}\right)$ where $\tilde{g}^{i}(\alpha, \theta^{i}, \theta^{j}), \tilde{g}^{j}(\alpha, \theta^{j}, \theta^{i})$ is the Nash equilibrium. The policy choices adapt to regional preferences, but fail to account for the spillover. Straightforwardly, the public good will be under-consumed in a Nash equilibrium when $\alpha \in (0, 1)$.

3. Evaluating relative welfare

In this section, we formally revisit the question of how consumption spillovers influence the relative merit of (de)centralization. The optimal policy choices (1) and (2) may confirm the predominant view that demand for centralization widens as the consumption spillover becomes more pronounced. To assess the validity of this reasoning, we consider preferences to be

$$v(G) = \begin{cases} \frac{1}{1-\eta} G^{1-\eta} & \eta \in R^{+} \setminus \{1\}, \\ \ln G & \eta = 1. \end{cases} \tag{3}$$

$\eta$ is the elasticity of the marginal utility of public consumption, $-v''(G) \frac{G}{v'(G)}$. The simplification is adopted for expositional clarity as it allows for a transparent and tractable characterization of how the curvature of $v(G)$ influences relative welfare.

The first-order conditions (1) and (2) yield as closed-form solutions of the equilibrium public expenditure levels

$$\tilde{g} = (\bar{\theta}(1 + \alpha))^{\frac{1}{\eta}} \frac{1}{1 + \alpha}$$

and

$$\tilde{g}^{i} = (\theta^{i})^{\frac{1}{\eta}} - \alpha(\theta^{j})^{\frac{1}{\eta}} \frac{1}{1 - \alpha^{2}}, \quad i \neq j. \tag{4}$$

The contribution level of the low-preference region, $\tilde{g}^{i}$, may violate the non-negativity constraint. The condition guaranteeing $\tilde{g}^{i} > 0$ is

$$\alpha < \left(\frac{\theta^{i}}{\bar{\theta}^{h}}\right)^{\frac{1}{\eta}}. \tag{5}$$

With decentralized policy-making region $i$’s utility change when spillovers magnify is

$$\frac{\partial u^{i}}{\partial \alpha} = \frac{\partial u^{i}}{\partial g^{i}} \frac{\partial g^{i}}{\partial \alpha} + \frac{\partial u^{i}}{\partial \tilde{g}^{i}} \frac{\partial \tilde{g}^{i}}{\partial \alpha} + \frac{\partial u^{i}}{\partial \tilde{g}^{j}} \frac{\partial \tilde{g}^{j}}{\partial \alpha}, \quad i \neq j. \tag{6}$$

The first term describes region $i$’s utility change due to the adjustment in its contribution to the public good. Invoking the envelope theorem, the welfare effect vanishes. The second term captures a utility gain. Keeping

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The first term describes region $i$’s utility change due to the adjustment in its contribution to the public good. Invoking the envelope theorem, the welfare effect vanishes. The second term captures a utility gain. Keeping
contribution levels constant, region $i$ is able to benefit more amply from region $j$’s public expenditures as $\alpha$ increases. Important for the subsequent analysis, the beneficial effect is more pronounced the larger the level of public expenditure in region $j$. The last term reflects how region $j$’s adjustment in the contribution to the public good influences region $i$’s welfare. Inserting $\partial \tilde{u}^i/\partial \alpha = \theta^i v' (\tilde{G}^i) \tilde{g}^i$, $\partial \tilde{u}^i/\partial \tilde{g}^i = \alpha$ and, from (2), $v' (\tilde{G}^i) = 1/\theta^i$ into (6), the change in region $i$’s utility simplifies to

$$\frac{d \tilde{u}^i}{d \alpha} = \tilde{g}^i + \alpha \frac{d \tilde{g}^i}{d \alpha}, \quad i \neq j. \tag{7}$$

Using (7), the change in welfare under decentralization $W^d = \tilde{u}^h + \tilde{u}^l$ is

$$\frac{d W^d}{d \alpha} = \tilde{g}^l + \tilde{g}^h + \alpha \left( \frac{d \tilde{g}^l}{d \alpha} + \frac{d \tilde{g}^h}{d \alpha} \right) \tag{8}$$

where the last equation follows from differentiating $\tilde{g}^l$ and $\tilde{g}^h$ w.r.t. $\alpha$ (see (4)), inserting the derivatives into the first equation, and rearranging terms.

The effect of a rise in spillovers on welfare under decentralization $W^c = \tilde{u}^h + \tilde{u}^l$ is

$$\frac{d W^c}{d \alpha} = \frac{\partial (\tilde{u}^h + \tilde{u}^l)}{\partial \tilde{g}} \frac{d \tilde{g}}{d \alpha} + \frac{\partial (\tilde{u}^h + \tilde{u}^l)}{\partial \alpha} \cdot \frac{d \tilde{g}}{d \alpha} \tag{9}$$

The first term reflects the virtue of centralization to account for spillovers. A higher spillover renders public consumption more valuable from a social perspective and, in consequence, the central government adjusts spending levels. Applying the envelope theorem, the policy response proves neutral for aggregate welfare. The remaining term represents a welfare gain. For given expenditure levels, both regions can more amply benefit from the neighbor region’s public expenditures in response to larger spillovers. Thus, the utility rise is larger the larger the expenditure level chosen by the central government. Inserting $\partial \tilde{u}^i/\partial \alpha = \theta^i v' (\tilde{G}) \tilde{g}$ and, from (1), $v' (\tilde{G}) = 1/\theta (1 + \alpha)$, the welfare change (9) is

$$\frac{d W^c}{d \alpha} = \frac{2}{1 + \alpha} \tilde{g}. \tag{10}$$

Combining (8) and (10), the change in the welfare differential $W = W^c - W^d$ becomes

$$\frac{d W}{d \alpha} = \frac{1}{1 + \alpha} \left( 2 \tilde{g} - (\tilde{g}^h + \tilde{g}^l) \right). \tag{11}$$

Hence, the difference between welfare under centralization and decentralization relates to the difference in public outlays under both modes of fiscal decision-making. It increases (decreases) if, and only if, public expenditures under centralization, $2 \tilde{g}$, exceed (fall short of) public expenditures under decentralization, $\tilde{g}^h + \tilde{g}^l$.

Expenditure levels will most notably depend on the curvature of the utility function $v(G)$ (parameterized by $\eta$) and the amount of spillovers (parameterized by $\alpha$). As shown in the appendix, decentralized policy choices yield a higher level of public expenditures if, and only if, spillovers and the elasticity of marginal utility are sufficiently small. Formally,

**Lemma.** Public expenditures satisfy $2 \tilde{g} < \tilde{g}^h + \tilde{g}^l$ if and only if $\alpha \in [0, \alpha^*)$, $\alpha^* \in (0, (\theta^l/\theta^h)^{1/2}]$, and $\eta < 1$. For all other admissible combinations of $\alpha$ and $\eta$, public expenditures satisfy $2 \tilde{g} \geq \tilde{g}^h + \tilde{g}^l$.

Intuitively, when spillovers are small, the insensitivity of central policy to local preferences yields a level of expenditure which lies in between the levels the low-type region and high-type region choose non-cooperatively, i.e. $\tilde{g}^l < \tilde{g} < \tilde{g}^h$. Furthermore, when $v' (G)$ does not drop too fast (i.e. $\eta < 1$) and thereby the demand for public consumption is sufficiently elastic, the expenditure level, which a high-type region selects non-cooperatively, significantly exceeds the uniform level, $\tilde{g}^h \gg \tilde{g}$. Aggregate expenditures are consequently higher in the uncoordinated equilibrium, $\tilde{g}^h + \tilde{g}^l > 2 \tilde{g}$. In all other cases, centralized policy choices yield weakly higher levels of public spending.9

Recalling (11), we can state:

**Proposition 1.** Assume preferences for public consumption to be iso-elastic.

(i) For $\eta < 1$ a marginal rise in spillovers decreases (increases) the welfare differential $W$ provided spillovers are small (large).

(ii) For $\eta \geq 1$ a marginal rise in spillovers weakly increases the welfare differential $W$ independently of the magnitude of spillovers.

Fig. 1 illustrates the set of $\alpha$-values which yields a negative slope of $W$ with respect to $\alpha$. The underlying preference parameters are $\theta^l = 2$ and $\theta^h = 7$. For all

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9 The qualification “weakly” refers to the generic case of logarithmic utility ($\eta = 1$) and the absence of spillovers ($\alpha = 0$). For this parameter combination, both modes of fiscal decision-making are equivalent in terms of aggregate spending.
combinations \((\alpha, \eta)\) above the thin, upward-sloping line both regions choose a positive contribution to the public good. The solid, downward-sloping line partitions the space according to the sign of \(\frac{dW}{d\alpha}\). Note, the figure is restricted to \(\eta \leq 1\). For \(\eta > 1\) the sign of \(\frac{dW}{d\alpha}\) is unambiguously positive (independent of the magnitude of \(\alpha\))—see part (ii) of Proposition 1.

At this point, it might be informative to relate the paper more firmly to the literature on (de)centralization. Existing analyses use preference functions which tend to suppress the identified non-monotonicity of the welfare difference—either because preferences lack concavity or they are logarithmic in public consumption. For instance, Lockwood (2002) and Rubinchik-Pessach (2004) assume additively linear preferences \((\eta = 0)\) in their analysis of discrete public good provision. Continuous solutions to the governments’ optimization problems do not exist for this class of preferences which precludes the computation of \(\frac{dW}{d\alpha}\). Differently, Besley and Coate (2003) use quasi-linear, logarithmic utility \((\eta = 1)\). With uniform central policies, the non-monotonicity of \(W\) in \(\alpha\) evaporates for this type of preferences—see part (ii) of Proposition 1.

From a policy perspective, a crucial question is whether the non-monotonicity of the welfare difference \(W\) yields a non-monotone sign of \(W\), i.e. whether \(W\) changes only at some positive level of spillovers. (13) shows that \(W\) slopes upward at any switching point, changing its sign from negative to positive. Given that \(W\) is continuous in \(\alpha\),\(^{10}\) the \(W\)-curve crosses the 0-line at most once. Consequently, the sign of \(W\) is monotonic.

Furthermore, we can show that a negatively-sloped welfare differential \(W\) does not exist when \(W > 0\). Inserting (4) into (11)

\[
\frac{dW}{d\alpha} \bigg|_{W=0} = \frac{1}{1+\alpha \eta} \left(\tilde{\vartheta}^h + \tilde{\vartheta}^l\right).
\]

The term in curly brackets is increasing in \(\alpha\). Thus, if \(W\) slopes positively at some value of \(\alpha\), it also slopes positively for larger values of \(\alpha\). Combining the finding with the result that \(\frac{dW}{d\alpha} > 0\) at \(W = 0\), we can conclude that \(\frac{dW}{d\alpha} > 0\) whenever \(W > 0\). Summarizing the results:

**Proposition 2.** A rise in the amount of spillovers may decrease the welfare difference \(W\) only when decentralization is the optimal mode of fiscal governance, i.e.

\(\tilde{g}\) is continuous in \(\alpha\) and \(\tilde{g}^l\) and \(\tilde{g}\) are continuous in \(\alpha\)—see Eq. (4). Thus, the welfare differential (12) also varies continuously with \(\alpha\).
\(W < 0\). Thereby, it will not justify a switch from centralization to decentralization.

Importantly, the preceding discussion shows that the non-monotonicity of \(W\) does not invalidate the bottom line of the “Decentralization Theorem” that centralization (decentralization) is welfare-superior when spillovers are sufficiently high (low).

4. Concluding comments

The paper provides a formal treatment of how relative welfare with (de)centralized policy relates to the strength of spillovers in public consumption. Most of the discussion on the costs and benefits of fiscal federalism rests on a welfare trade-off which is taken to be monotone in the primitive of the economy. In contrast to the presumption, the analysis points to a non-monotone trade-off. A marginally higher degree of spillovers may promote the well-being of constituents under decentralization compared with centralization.

The analysis reveals that a non-monotonicity of \(W\) will only arise when decentralization is welfare-enhancing. The finding may not extend to models of fiscal federalism which differ from the specification adopted in the paper. Suggestively, a non-monotone sign of \(W\) may arise in models in which decentralization generates distortions beyond the failure to internalize spillovers or in which centralization exhibits allocative advantages in addition to the internalization of spillovers. In these cases, the \(W\)-curve potentially shifts upward and multiple crossing points with the 0-line may exist. We leave a rigorous analysis of the reasoning to future research.

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Appendix A. Proof of Lemma

Following (4) and recalling the definition \(\bar{\theta} := (\theta^h + \theta^l)/2\), we can write

\[
(2\bar{g} - (\bar{g}^h + \bar{g}^l)) \bigg|_{\alpha = 0} = 2\left(\frac{(\theta^h + \theta^l)}{2}\right)^{\frac{1}{\eta}} - \left((\theta^h)^{\frac{1}{\eta}} + (\theta^l)^{\frac{1}{\eta}}\right).
\]  

(15)

We next determine the sign of the expenditure differential for \(\eta < 1, \eta > 1\) and \(\eta = 1\) separately.

Consider \(\eta < 1\). For \(\eta < 1\) the function \((\cdot)^{\frac{1}{\eta}}\) is convex. Hence, (15) implies

\[
(2\bar{g} - (\bar{g}^h + \bar{g}^l)) \bigg|_{\alpha = 0} < 0.
\]

As (14) is increasing in \(\alpha\), there exists a threshold \(\alpha^* \in (0, (\theta^l/\theta^h)^{\frac{1}{\eta}}]\) with \(2\bar{g} < \bar{g}^h + \bar{g}^l\), \(\forall \alpha \in [0, \alpha^*]\).

Consider \(\eta > 1\). Rearranging (14)

\[
2\bar{g} - (\bar{g}^h + \bar{g}^l) = \frac{1}{1 + \alpha} \left\{ 2(1 + \alpha)^{\frac{1}{\eta}} \left(\frac{(\theta^h + \theta^l)}{2}\right)^{\frac{1}{\eta}} - \left((\theta^h)^{\frac{1}{\eta}} + (\theta^l)^{\frac{1}{\eta}}\right) \right\}.
\]

(16)

For \(\eta > 1\) the function \((\cdot)^{\frac{1}{\eta}}\) is concave. Hence, given by Eq. (15), the difference in expenditure levels satisfies

\[
(2\bar{g} - (\bar{g}^h + \bar{g}^l)) \bigg|_{\alpha = 0} > 0.
\]

(17)

The term \(2(1 + \alpha)^{\frac{1}{\eta}}\) in (16) is increasing in \(\alpha\). Following (16) and (17), we can conclude that \(2\bar{g} - (\bar{g}^h + \bar{g}^l) > 0, \forall \alpha \in [0, (\theta^l/\theta^h)^{\frac{1}{\eta}}]\).

Consider \(\eta = 1\). Given by (15), we have \(2\bar{g} = \bar{g}^h + \bar{g}^l\) at \(\alpha = 0\). Furthermore, the expression in (14) is increasing in \(\alpha\). Consequently, we get \(2\bar{g} - (\bar{g}^h + \bar{g}^l) > 0, \forall \alpha \in [0, (\theta^l/\theta^h)^{\frac{1}{\eta}}]\).

Combining the results derived for \(\eta < 1, \eta > 1\) and \(\eta = 1\) completes the proof.

References