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# An analytical model for the formation of economic clusters ☆

Ludwig Chincarini [a], Neer Asherie [b,*]

[a] Department of Economics, Pomona College, 425 N. College Avenue #208, Claremont, CA 91711, United States
[b] Department of Physics, Yeshiva University, Belfer Hall 1412, 2495 Amsterdam Avenue, New York, NY 10033-3312, United States

**Abstract**

A simple spatial economy derived from microeconomic foundations is presented to gain insight into the formation of economic clusters. In this model, the formation of economic clusters is a consequence of the competition between economic forces that are consistent with atomistic agents maximizing their utility. An analytic approach is used to obtain the evolution of economic clusters. With this approach, the number of clusters which will grow can be predicted. These results are derived in the traditional one-dimensional geometry and extended to the more realistic two-dimensional landscape.


*JEL classification:* R1; C6; F12
*Keywords:* Economic geography; Economic clusters; Agglomeration; Mathematical modelling; City formation

## 1. Introduction

Since the Industrial Revolution when large scale urbanization began, there has been an investigation of the factors that shape the economic landscape.[1] Beginning with Von Thünen (Von Thünen, 1826, 1966), models have been proposed to explain the regular ordering of economic structures (Christaller, 1933, 1966; Losch, 1967; Heller, 1993; Knox and Agnew, 1994; Makse et al., 1995; Quah, 2002). One recent approach is that of Krugman (Krugman, 1991, 1994) in which it is the competition between two opposing forces in the economy—increasing returns to scale in manufacturing goods and costs to distributing goods to immobile consumers of the economy—that leads to city formation. In addition, several authors (Chincarini, 1995; Henderson and Mitra, 1995; Krugman, 1995) have begun to study the formation of new urban centers, called Edge Cities, which have emerged predominantly across North America, in the past three decades (Garreau, 1991).

Though previous models have contributed many important ideas to the field of economic geography, their utility when studying the evolution of clusters has been limited by at least one of three main factors; (i) inadequate economic

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* Corresponding author. Tel.: +1 212 960 5452.
*E-mail addresses:* chincarinil@hotmail.com (L. Chincarini), asherie@yu.edu (N. Asherie).
[1] "Much of urban theory developed during the past 50 years has been unable to link the underlying economic and ecological theory of cities to the actual spatial patterns which we observe… A synthesis is in the offing, and it looks as though the time is ripe now for the new approach to cities and urban form for which we have been waiting for more than a generation" (Batty, 1995).

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foundations, e.g. diffusion-limited aggregation theories (Makse et al., 1995); (ii) inappropriate or unrealistic shapes of the urban landscape e.g. finite and infinite straight lines (Krugman, 1994; Henderson and Mitra, 1995; Fujita and Mori, 1995)[2]; and (iii) the absence of analytical solutions to predict the number of economic clusters e.g. simulation work (Krugman, 1991, 1995). Thus, although many models of cluster evolution have been presented in the literature, there has not been a complete mathematical analysis of cluster formation for even the simplest type of model based on microeconomic foundations.[3]

In order to address some of the outstanding questions regarding cluster formation, we present in this paper a complete analysis of the evolution of economic clusters for a simple economic model of the economy. We show using analytic methods, how the number of clusters that form is related to the underlying microeconomic forces. We first examine a one-dimensional geometry which has been partially studied in the literature. We rigorously derive results that were previously obtained "intuitively" or through the use of simulations. We then present a solution for the heretofore unsolved two-dimensional model of the economy.[4] We show how novel structural features are observed in the two-dimensional geometry.

To begin our analysis, we consider an economy that consists of mobile workers and immobile farmers. There are increasing returns in producing manufactured goods and constant returns to scale in producing farm products. Everyone has tastes for all goods, i.e. Dixit–Stiglitz preferences are assumed. We assume, following Krugman (1991), that there are two forces acting on the spatial structure: increasing returns to scale causing economic activity to agglomerate, and costs to distributing goods to the immobile farmers causing economic activity to disperse.

In our analysis we study the evolution of the economy from a uniformly distributed spatial structure of workers and farmers. This uniform distribution of workers is slightly perturbed into an arbitrary distribution of workers over the spatial structure. Eventually we observe a regular clustering of economic activity in the economy, which we associate with city formation. The distribution of workers which emerges depends on the transportation costs, economies of scale, the total land area and the fraction of workers in the economy.

As stated earlier, two models of the spatial structure are presented. The one-dimensional model consists of an economy which lies along the circumference of a circle. A more realistic two-dimensional model is also presented which describes the process of economic cluster growth on the surface of a sphere. Both models predict that regularly spaced clusters will form in the spatial structure and are consistent with the empirical work of others (Christaller, 1933; Szymanski and Agnew, 1981). Given the transportation costs, the economies of scale, the relevant land area, and the fraction of workers in the economy, we can predict analytically (without simulations) the number of clusters that will form in our one- and two-dimensional models. Although the same microeconomic factors operate in both models, the two-dimensional model allows us to explicitly see the types of configurations economic clusters may assume in the real world. Our two-dimensional model thus provides an attempt to bridge between theoretical and empirical work on cluster formation. In particular, the two-dimensional model allows for a sufficiently rich spatial structure which could include Edge Cities.

The paper is organized as follows: Section 2 introduces the one-dimensional spatial model and the various techniques used in the analysis of cluster formation; Section 3 discusses how the parameters of the model affect the distribution of economic clusters; Section 4 presents a two-dimensional model of cluster formation; Section 5 discusses the two-dimensional model; a possible connection with Edge Cities and suggestions for further research are presented in Section 6; and Section 7 concludes the paper.

## 2. The one-dimensional model

### 2.1. The core-periphery economy

For completeness, we begin by describing the core elements of the basic core-periphery model (Krugman, 1991). We assume an economy with two sectors, manufacturing and agriculture. All consumers have demand for both types of goods, that is

$$U = C_{\mathrm{M}}^{\mu} C_{\mathrm{A}}^{1-\mu}. \tag{1}$$

[2] This deficiency has been recently highlighted by Neary (2001) who wrote that most models have the drawback that "…space is almost always one-dimensional..." and he goes on to say that "perhaps it will prove possible to extend the Dixit–Stiglitz approach to a two-dimensional plane…".

[3] Forslid and Ottaviano (2003) have produced an analytically tractable model for two regions. Other analytical models have departed significantly from the original core-periphery models (Baldwin, 1999; Harrigan and Venables, 2006; Norman and Venables, 2004; Tabuchi et al., 2005).

[4] Quah (2002) presents another type of model based upon knowledge accumulation which depicts geographical dispersion on the surface of a globe.

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where $C_{\mathrm{M}}$ is the quantity of the manufacturing good composite, $C_{\mathrm{A}}$ is the quantity of the agricultural good, and $\mu$ is the share of expenditure on manufactured goods. We assume that there is a single, homogeneous agricultural good. The manufactured good is a composite of a large number of product varieties with a constant elasticity of substitution (CES), $\sigma$, between any two goods, thus

$$C_{\mathrm{M}} = \left[\sum_i c_i^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}. \tag{2}$$

where $c_i$ is the consumption of manufacturing good variety $i$. We make the simplifying assumption that labor is the only factor in production. There are two types of labor used in production: laborers that produce manufactured goods (workers) and laborers that produce agricultural goods (farmers). It is assumed that workers are mobile, while farmers are not.

We consider a continuous economy located on a circle of radius $r$. The workers and farmers are distributed along the circumference of the circle. In particular, we will let $\lambda(\phi, t)$ denote the density of workers along the circle at any time $t$ and $\psi(\phi)$ denote the density of farmers along the circumference of the circle, with $\phi$ the angle around the circle from 0 to $2\pi$ radians (see Fig. 1). Since the farmers are immobile, $\psi$ is independent of time. In our one-dimensional economy the variable for distance is $\phi$. The radius $r$ is only a parameter.

We assume that farming takes place under constant returns to scale. Therefore the quantity of an agricultural good $Q_{\mathrm{A}}(\phi)$ produced at a specific location $\phi$ is equal to the agricultural labor $L_{\mathrm{A}}(\phi)$ at the same location:

$$Q_{\mathrm{A}}(\phi) = L_{\mathrm{A}}(\phi). \tag{3}$$

Note that neither the quantity of agricultural goods nor the agricultural labor depend on time as the farmers are immobile.

Manufacturing production of good $i$ involves economies of scale. Therefore the quantity of the $i$th manufacturing good $Q_{\mathrm{M}i}(\phi, t)$ is related linearly to the manufacturing labor $L_{\mathrm{M}i}(\phi, t)$ for the same good:

$$Q_{\mathrm{M}i}(\phi, t) = a + bL_{\mathrm{M}i}(\phi, t), \tag{4}$$

where $a$ and $b$ are constant coefficients representing the fixed costs and marginal costs of production respectively.

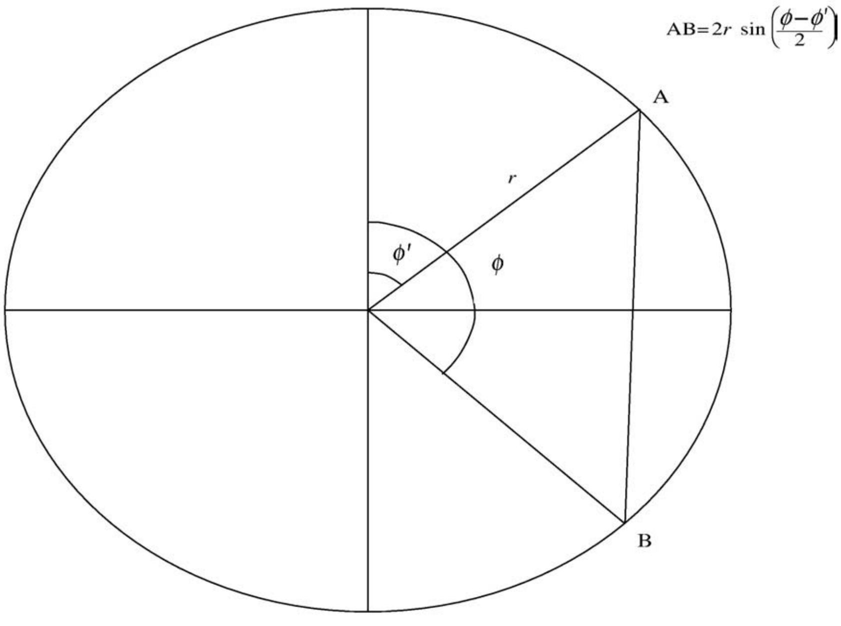


Fig. 1. The one-dimensional economy located on a circle.

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Let $L_A$ and $L_M$ represent the entire supply of farmers and workers in the economy. The density of farmers and workers are then given by

$$\psi(\phi) = L_A(\phi)/L_A$$
$$\lambda(\phi,t) = \sum_i L_{Mi}(\phi,t)/L_M \tag{5}$$

At any point in time, there will be local full employment.

Next, we introduce a cost to transporting manufactured goods. For simplicity, we assume that there is no cost to transporting agricultural goods. Our transport costs are a modified iceberg form. That is, if $x$ goods are shipped from $\phi$ to $\phi'$, then only $z$ goods arrive, where $z = xe^{-\tau r[1-\cos(\phi-\phi')]}$. Our transportation costs are included as $\tau$ and our distance function between locations is represented by $D(\phi, \phi') \equiv r[1-\cos(\phi-\phi')]$. This factor incorporates the idea that the further apart two locations are, the higher is the transportation cost for shipping goods between the two locations. It differs from the "iceberg" model (Krugman, 1991) as the exponent does not represent the distance between the two locations exactly, but does so proportionately.[5] Our form of the transportation cost factor takes into account the circular geometry: goods are always shipped via the shorter route between two points. It is precisely because we are dealing with a closed geometry and have chosen an appropriate transportation cost factor that we are able to obtain analytical solutions for the evolution of the economy. Observing Fig. 1, one will notice that a proxy for the actual distance around the circle between locations A and B is the straight line drawn which connects them. The length of this line is $2r\sin\left(\frac{\phi-\phi'}{2}\right)$. The straight line measure of the distance cannot be used directly for it changes sign when $\phi'$ is greater than $\phi$. In order to obtain a distance measure that is rotationally invariant, we simply square our sine term.[6] This is the distance we use in our transportation cost factor. We actually write $r[1-\cos(\phi-\phi')]$ as our distance factor. This last form is identically equal to $2r\sin^2[(\phi-\phi')/2]$, but is more easily generalized to the two-dimensional case, which we discuss later.

We denote the number of workers and farmers by $\mu$ and $1-\mu$ respectively.[7] All prices and wages will be measured in terms of the agricultural good. Given this setup, we can express the income $Y(\phi, t)$ at each location in terms of the equilibrium wage rate $w(\phi, t)$ (which is the worker wage in terms of the agricultural good wage) as

$$Y(\phi,t) = [(1-\mu)\psi(\phi) + \mu\lambda(\phi,t)w(\phi,t)]. \tag{6}$$

This equation, which represents total income at each location relative to the agricultural wages, can be understood as follows. The farmer income at location $\phi$ is $(1-\mu)\psi(\phi)w_A$ and worker income is $\mu\lambda(\phi, t)w_M$, where $w_A$ and $w_M$ are the nominal farmer and worker wages respectively. If we define all prices and wages in terms of the agricultural good, then the income in each location is obtained by dividing the nominal farmer and worker incomes by $w_A$, giving Eq. (6).

Next, we turn to the true or ideal price index of the manufactures composite to consumers at each location. In the standard Dixit–Stiglitz monopolistic competition framework (Dixit and Stiglitz, 1977), the price index is given by $\left(\sum_{i=1}^{n} p_i^{1-\sigma}\right)^{\frac{1}{1-\sigma}}$, where $p_i$ is the price of each good $i$ being produced. The profit maximization condition in the Dixit–Stiglitz framework results in an equilibrium where all varieties are produced for the same price, $p$, which is given by $p_i = p = wb\frac{\sigma}{\sigma-1}$. In order to simplify the exposition, we choose units of $b$ so that $p = w\left(\text{i.e. } b = \frac{\sigma-1}{\sigma}\right)$. Thus, the prices of manufactured goods at each location $\phi$ are equal to manufacturing wages. However, in order to construct a price index in our geographical economy, we must adjust manufactured good prices for transportation. Thus, for each unit of a manufactured variety to make it from $\phi'$ to $\phi$, then $e^{\tau r[1-\cos(\phi-\phi')]}$ must be shipped, so the total price[8] on arrival is $w(\phi')e^{\tau r[1-\cos(\phi-\phi')]}$. Thus, we substitute this term in for $p$ in the Dixit–Stiglitz price index. We must also modify the

[5] For a discussion of iceberg transport cost functions, see McCann (2005) and Combes and Lafourcade (2005).

[6] The form for the transportation cost factor must be rotationally invariant for we assume that the geography of our economy is uniform i.e., there are no mountains or rivers which make one route more expensive than another; only the absolute distance matters. We have chosen a specific form for the transportation cost factor that allows us to obtain analytical solutions. Other forms are equally valid, but for most choices simulations would be required to calculate the number of clusters that emerge.

[7] This normalization will set economy-wide income to 1. One can also think of this as the relative supply of workers and farmers. That is, $\frac{\mu}{1-\mu} = \frac{L_M}{L_A}$. Then the income equation would essentially be income of a region divided by the laborer population.

[8] One can think of this as the cost, insurance, and freight (c.i.f.) price.

price index as to account for goods shipped from all locations in proportion to the manufacturing labor force in that location. With these modifications, the true price index of manufactured goods $T(\phi, t)$ at location $\phi$ is

$$T(\phi,t)=\left[\int_0^{2\pi}\lambda(\phi',t)w(\phi',t)^{1-\sigma}e^{-\tau r(\sigma-1)[1-\cos(\phi-\phi')]}rd\phi'\right]^{\frac{1}{1-\sigma}}. \tag{7}$$

Given these price indices, we can solve for the equilibrium wage rates in this economy at any time $t$. The short-run equilibrium wage in any location will depend on demand and supply forces for those particular goods from other regions around the geographical economy. The equilibrium wage rate can be derived from Eq. (7) as shown in Krugman (1991) for a discrete space system. A translation of this derivation to our model gives the equilibrium wage as

$$w(\phi,t)=\left[\int_0^{2\pi}Y(\phi',t)T(\phi',t)^{\sigma-1}e^{-\tau r(\sigma-1)[1-\cos(\phi-\phi')]}rd\phi'\right]^{\frac{1}{\sigma}}. \tag{8}$$

This equilibrium wage rate depends on a weighted sum of purchasing power at all locations through $Y(\phi, t)$. That is, higher income at other locations implies a higher equilibrium wage rate at location $\phi$, but this impact is smaller the further away the other location is. In addition, the equilibrium wage depends on the true price index, $T(\phi, t)$. This reflects the competition from other producers in other locations. If the true price index at location $\phi'$ is lower, all else equal, the equilibrium wage will be lower at location $\phi$ due to competition from producers in other regions around the circle.

So far we have determined wage rates, $w(\phi, t)$, in terms of the agricultural good. Workers will care about real wages with respect to all goods, including manufactured goods. The real wage of workers, $\omega(\phi)$ is given by

$$\omega(\phi,t)=w(\phi,t)T(\phi,t)^{-\mu}. \tag{9}$$

Given Eqs. (6)–(8), we can solve for $Y(\phi, t)$, $T(\phi, t)$, and $w(\phi, t)$ in terms of our economy-wide parameters. These in turn determine our real wages, $\omega(\phi, t)$.

The final piece of this model is to introduce a mechanism by which the mobile workers decide to move from one location on the circle to another. We make the assumption that workers move to locations that offer higher real wages.[9] That is, the distribution of workers in the economy at time $t$ evolves according to

$$\frac{\partial\lambda(\phi,t)}{\partial t}=\rho[\omega(\phi,t)-\bar{\omega}(t)]\lambda(\phi,t), \tag{10}$$

where $\rho$ is a constant rate parameter, $\omega(\phi, t)$ is the real wage at location $\phi$ at time $t$ and $\bar{\omega}(t)$ is the average real wage at time $t$. This last quantity is given by

$$\bar{\omega}(t)=\int_0^{2\pi}\lambda(\phi,t)\omega(\phi,t)rd\phi. \tag{11}$$

Intuitively, Eq. (10) says that workers move towards regions with above-average real wages and move away from regions with below-average real wages.

Now that we have developed the economy of our temporal geography, we rewrite the entire economy as

$$Y(\phi,t)=\alpha[(1-\mu)\psi(\phi)+\mu\lambda(\phi,t)w(\phi,t)], \tag{12}$$

$$T(\phi,t)=\left[\beta\int_0^{2\pi}\lambda(\phi',t)w(\phi',t)^{1-\sigma}e^{-\tau r(\sigma-1)[1-\cos(\phi-\phi')]}rd\phi'\right]^{\frac{1}{1-\sigma}}, \tag{13}$$

$$w(\phi,t)=\left[\delta\int_0^{2\pi}Y(\phi',t)T(\phi',t)^{\sigma-1}e^{-\tau r(\sigma-1)[1-\cos(\phi-\phi')]}rd\phi'\right]^{\frac{1}{\sigma}}, \tag{14}$$

[9] Farmers are immobile and thus are not a concern in the dynamic geographical economy.

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$$\omega(\phi,t) = w(\phi,t)T(\phi,t)^{-\mu}, \tag{15}$$

$$\varpi(t) = \int_0^{2\pi} \lambda(\phi,t)\omega(\phi,t)rd\phi, \tag{16}$$

$$\frac{\partial\lambda(\phi,t)}{\partial t} = \rho[\omega(\phi,t) - \varpi(t)]\lambda(\phi,t), \tag{17}$$

where $\alpha = 2\pi r$, $\beta = \frac{1}{\Gamma_0(\tau r(\sigma-1))}$ and $\delta = \frac{1}{2\pi r\Gamma_0(\tau r(\sigma-1))}$. The function $\Gamma_0$ is defined in the Appendix, Section A.4. The normalization factors $\alpha$, $\beta$, and $\delta$ have been introduced so that the equilibrium value of most quantities is one (see the next section).

In this core-periphery model of the economic geography, the exogenous variables of the system are the distribution of farmers, $\psi(\phi)$, and workers, $\lambda(\phi, t)$.

## 2.2. The evolution of spatial patterns

We wish to study the evolution of the economy from equilibrium. First we will find an initial equilibrium for the economy. We will then perturb the economy slightly and observe how it changes from the equilibrium state. Explicitly, we assume that each dynamic variable of the economy can be separated into a time-independent equilibrium component and a perturbed part, which is taken to be small compared to the equilibrium component. Thus, we write the density of workers as $\lambda(\phi, t) = \lambda_0(\phi) + \lambda_1(\phi, t)$ with $|\lambda_1(\phi, t)| \ll |\lambda_0(\phi)|$, and similarly for the other variables. This technique is known as perturbation theory or linear stability theory (more details may be found in the Appendix, Section A.1; see also Reichl, 1998). Note that, by assumption, the perturbation must be small. Therefore our analysis is limited to the initial stages of the evolution of the economy.

The motion of a ball on different surfaces provides a conceptual example of perturbation theory. If a ball is placed in a deep trough, it will not move; the ball is in equilibrium. The ball is then tapped lightly i.e., it is perturbed from its equilibrium position. The ball will then begin to oscillate inside the trough, but will never be far from its original position; the equilibrium is stable (in fact, due to friction, the ball will eventually return back to its starting point). Now imagine the ball is on the top of a hill. Again, it is stationary and therefore initially in equilibrium. However, in this case, if the ball is tapped, it will begin to roll down the hill and move further and further away from its initial position. In other words, the equilibrium is unstable. For the economy we are studying, if the equilibrium is stable, then after the perturbation the economy will return to its initial state. If the equilibrium is unstable, it will move away from its initial state and, as we will show, clusters will emerge.

From the equilibrium or zeroth order equations we find that $\lambda_0(\phi) = \frac{1}{2\pi r}$, $\psi_0(\phi) = \frac{1}{2\pi r}$, $Y_0(\phi) = 1$, $T_0(\phi) = 1$, $w_0(\phi) = 1$, and $\omega_0(\phi) = 1$. Thus, our economy is initially uniform in space; none of the equilibrium variables depends on $\phi$. Clearly $\omega_0 = \varpi_0$, i.e. the wage at each location at equilibrium is the same as the mean equilibrium wage. Given this equilibrium, we obtain the following expressions for the perturbed (first order) variables:

$$Y_1(\phi,t) = \alpha\mu\frac{w_1(\phi,t)}{2\pi r} + \alpha\mu\lambda_1(\phi,t), \tag{18}$$

$$T_1(\phi,t) = \frac{\beta}{1-\sigma}\int_0^{2\pi}\left[\lambda_1(\phi',t) + \frac{(1-\sigma)}{2\pi r}w_1(\phi',t)\right]e^{-\tau r(\sigma-1)[1-\cos(\phi-\phi')]}rd\phi', \tag{19}$$

$$w_1(\phi,t) = \frac{\delta}{\sigma}\int_0^{2\pi}[Y_1(\phi',t) + (1-\sigma)T_1(\phi',t)]e^{-\tau r(\sigma-1)[1-\cos(\phi-\phi')]}rd\phi', \tag{20}$$

$$\omega_1(\phi,t) = w_1(\phi,t) - \mu T_1(\phi,t), \tag{21}$$

$$\varpi_1(t) = \int_0^{2\pi}\left[\lambda_1(\phi,t) + \frac{\omega_1(\phi,t)}{2\pi r}\right]rd\phi, \tag{22}$$

$$\frac{\partial\lambda_1(\phi,t)}{\partial t} = \frac{\rho}{2\pi r}[\omega_1(\phi,t) - \varpi_1(t)]. \tag{23}$$

At this point, we want to observe how the landscape will change when we perturb the spatial structure from equilibrium. We will shortly see that, regardless of the initial perturbation, the economy, as described by the above equations, will eventually evolve into a distinct number of clusters. This optimal number of clusters is the one for which the forces causing agglomeration of workers (increasing returns) and the forces that disperse economic activity (transportation costs) offset each other. The number of clusters that will form depends on the parameters of the economy: transportation costs, $\tau$, economies of scale, $\sigma$, the relevant land area, $r$, and the fraction of workers in the economy, $\mu$. We now show how the number of clusters that will grow can be predicted analytically.

Whatever the perturbation of workers in the economy, we can represent it as a Fourier series expansion of the form (Arfken, 1970)

$$\lambda_1(\phi,t) = \sum_{m=-\infty;m\neq 0}^{m=\infty} \lambda_1 m(t) e^{im\phi} \tag{24}$$

where

$$\lambda_{1m}(t) = \frac{1}{2\pi}\int_0^{2\pi} e^{-im\phi}\lambda_1(\phi,t)d\phi. \tag{25}$$

Here $\lambda_{1m}(t)$ is the amplitude of the perturbation in the density of workers at time $t$. We exclude the $m=0$ Fourier harmonic because we are interested in perturbations which do not alter the total number of workers in the economy. Since we will always be assuming that $m\neq 0$, we will henceforth drop the explicit indication that $m\neq 0$ in our sums. The other perturbed quantities in Eqs. (18)–(23) may also be expressed as Fourier series analogous to Eq. (24): $\omega_1(\phi,t) = \sum_{m=-\infty}^{m=\infty} \omega_{1m}(t)e^{im\phi}$ etc.

An illustration of Fourier series can be given in terms of sound. The sounds we hear are usually made of many different frequencies. Decomposing a function as a Fourier series is equivalent to isolating the individual frequencies (harmonics) in a given sound.

It may be shown that the different harmonics (i.e. the terms with different values of $m$) in the Fourier series evolve independently. This follows from the linearity and rotational invariance of Eqs. (18)–(23) (see the Appendix, Section A.2 for an example). Thus, Eqs. (18)–(23) reduce to a set of linear equations for the time-dependent coefficients of the $m$th Fourier harmonic of the perturbed variables. For example, Eq. (23), which governs the time evolution of the economy, becomes

$$\frac{\partial \lambda_{1m}(t)}{\partial t} = \frac{\rho}{2\pi r}[\omega_{1m}(t)], \tag{26}$$

since, as shown in Section A.3 of the Appendix, $\bar{\omega}_1(t)=0$. By solving Eqs. (18)–(21) for the coefficients of the perturbed variables, Eq. (26) can be written as

$$\frac{\partial \lambda_{1m}(t)}{\partial t} = g_m \lambda_{1m}(t) \tag{27}$$

where $g_m$ is the growth rate of the $m$th Fourier harmonic of the perturbed worker density. Explicitly, $g_m$ is found to be

$$g_m = \rho\left[\frac{(1-\mu\eta)(\eta\mu-\eta^2)}{\sigma-\mu\eta-(\sigma-1)\eta^2} - \frac{\mu\eta}{1-\sigma}\right], \tag{28}$$

with

$$\eta = \frac{\Gamma_m(\tau r(\sigma-1))}{\Gamma_0(\tau r(\sigma-1))}, \tag{29}$$

where the function $\Gamma_m$ is defined in the Appendix, Section A.4.

The solution of Eq. (27) is

$$\lambda_{1m}(t) = \lambda_{1m}(t=0)e^{g_m t}. \tag{30}$$

Thus the amplitude of $m$th harmonic of the initial perturbation, $\lambda_{1m}(t)$, will grow if $g_m>0$.

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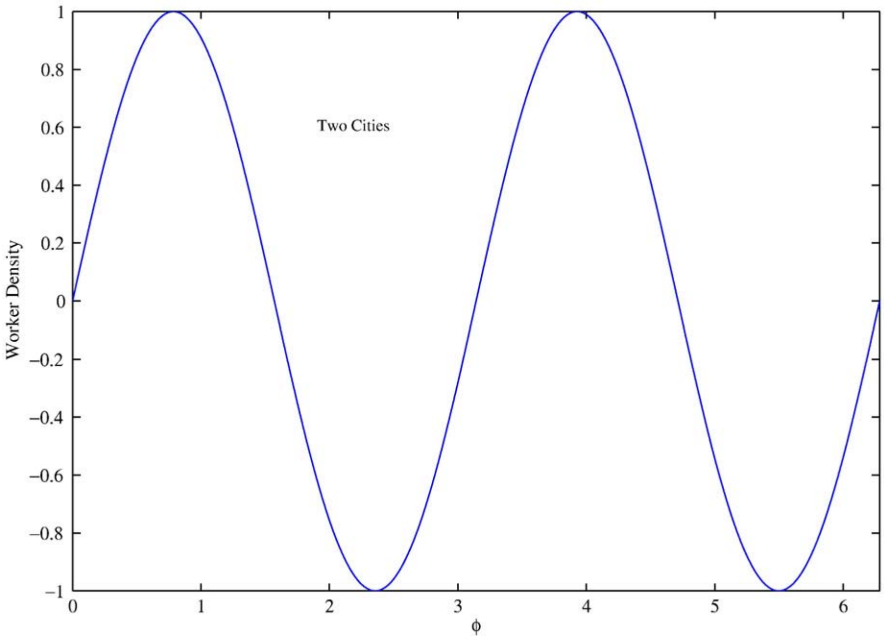


Fig. 2. One-dimensional spatial structure when two clusters form ($m^*=2$).

Since $\rho>0$, the condition for the growth of clusters requires that the quantity in the square brackets in Eq. (28) be positive. It is this growth rate of the perturbed distribution of workers that we investigate in the next section.

## 3. Cluster growth in one dimension

Eq. (30) shows how the $m$th harmonic of the perturbed density of workers grows in time. Using this equation together with Eq. (24), the total perturbation of the workers, $\lambda_1(\phi, t)$ is given by

$$\lambda_1(\phi,t)=\sum_{m=-\infty}^{m=+\infty}\lambda_{1m}(t)e^{im\phi}=\sum_{m=-\infty}^{m=+\infty}\lambda_{1m}(t=0)e^{g_m t}e^{im\phi}. \tag{31}$$

The sum on the right-hand side of Eq. (31) depends exponentially on time. Therefore, it will eventually be dominated by the term with the largest value of $e^{g_m t}$ i.e. by the term which has the largest growth rate (all other terms will be exponentially smaller). If we denote by $m^*$ the value of $m$ for which $g_m$ is a maximum, we see that eventually the distribution of workers becomes

$$\lambda_1(\phi,t)=\lambda_{1m^*}(t=0)e^{g_{m^*}t}e^{im^*\phi}, \tag{32}$$

where we have taken $m^*>0$ since the growth rates for $m^*$ and $-m^*$ are the same.[10]

Therefore, with time, out of our initial arbitrary perturbation, an ordered structure emerges. We associate the $m^*$th harmonic (the "dominant frequency") with the growth of $m^*$ clusters (see Fig. 2 for an illustration). We have assumed that only one term eventually dominates Eq. (32). The case where the maximum growth rate is the same for more than one $|m|$ is a set of measure zero and will not be dealt with here.

We now analyze the behavior of the growth rate $g_m$ and hence observe how the number of clusters which eventually form varies as the parameters of the economy are changed. Recall that the number of clusters which will eventually

[10] $g_{m*}$ can be viewed as the maximal eigenvalue of the real dynamical system described by Eqs. (12)–(17) subject to a small perturbation about equilibrium; see Reichl (1998).

Table 1
The growth rates $g_{m^*}/\rho$ at the dominant frequency $m^*$ for various $r$ and $\tau$

| | $\tau$=0.2 | $\tau$=0.4 | $\tau$=0.6 |
|---|---|---|---|
| $r$=1 | 0.0128 ($m^*$=1) | 0.0117 ($m^*$=2) | 0.0133 ($m^*$=2) |
| $r$=2 | 0.0117 ($m^*$=2) | 0.0111 ($m^*$=3) | 0.0134 ($m^*$=3) |
| $r$=3 | 0.0133 ($m^*$=2) | 0.0134 ($m^*$=3) | 0.0135 ($m^*$=4) |
| $r$=4 | 0.0111 ($m^*$=3) | 0.0128 ($m^*$=4) | 0.0125 ($m^*$=5) |

Recall that $m^*$ represents the number of clusters which form. Here $\mu$=0.2 and $\sigma$=4.

form, $m^*$, is the one for which the growth rate is a maximum. For convenience, we repeat here the expression for the growth rate

$$g_m = \rho\left[\frac{(1-\mu\eta)(\eta\mu-\eta^2)}{\sigma-\mu\eta-(\sigma-1)\eta^2} - \frac{\mu\eta}{1-\sigma}\right], \tag{33}$$

with

$$\eta = \frac{\Gamma_m(\tau r(\sigma-1))}{\Gamma_0(\tau r(\sigma-1))}. \tag{34}$$

We see that formally $g_m$ depends on four parameters: $r$, the radius of the circle, $\tau$, transportation costs per unit distance, $\sigma$, the elasticity of substitution, and $\mu$, the fraction of workers in the economy. Recall that $\rho$ is a positive constant (for all our numerical results we will present $g_m/\rho$). However, $r$ and $\tau$ always appear in the combination $r\tau$ (through $\eta$). Thus we may consider $g_m$ to be a function of only three parameters: $r\tau$, the total transportation cost, $\sigma$ and $\mu$. We now examine how the value of $m^*$ (the $m$ for which $g_m$ is a maximum) and hence the number of clusters which form changes as these parameters are varied.

We begin by considering the total transportation cost $r\tau$. We first study some simple limits. For $r\tau\to\infty$, $\eta_m\to 1$, and we find that $g_m/\rho\to(\sigma-1-\mu\sigma)/(1-\sigma)$. In other words, as the total transportation costs tend to infinity, clusters will not form unless $\mu>(\sigma-1)/\sigma$. With large transportation costs, the cost of transporting manufacturing goods from concentrated centers is very expensive. Therefore, only if the goods are sufficiently differentiated and there is a large enough manufacturing base, will clusters form. In the opposite limit of negligible transportation costs, $r\tau\to 0$,

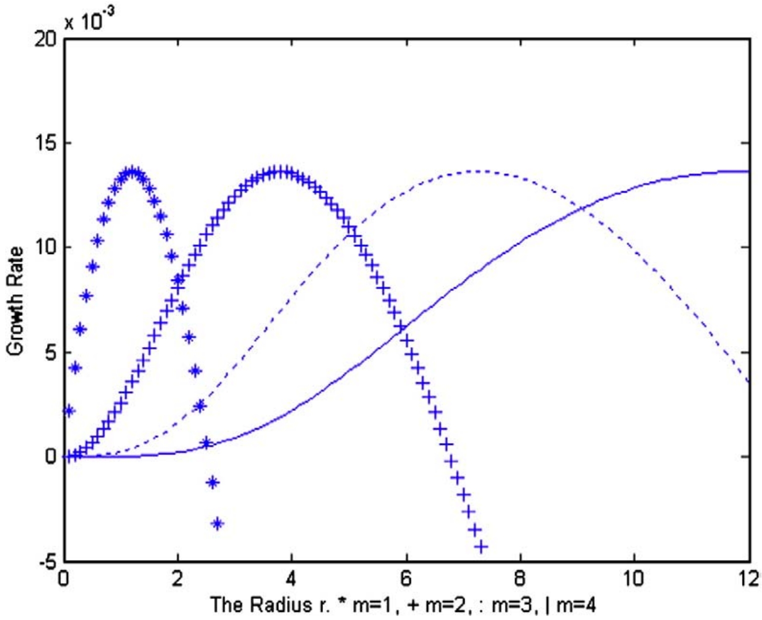


Fig. 3. The one-dimensional growth rate versus the radius $r$ for various $m$ ($\sigma$=0.4, $\tau$=0.2, and $\mu$=0.2).

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Table 2
The growth rates $g_{m^*}/\rho$ at the dominant frequency $m^*$ for various $r$ and $\sigma$

| | $\sigma=2$ | $\sigma=4$ | $\sigma=6$ |
|---|---|---|---|
| $r=1$ | 0.0249 ($m^*=1$) | 0.0128 ($m^*=1$) | 0.0059 ($m^*=2$) |
| $r=2$ | 0.0396 ($m^*=1$) | 0.0117 ($m^*=2$) | 0.0068 ($m^*=2$) |
| $r=3$ | 0.0448 ($m^*=1$) | 0.0133 ($m^*=2$) | 0.0080 ($m^*=3$) |
| $r=4$ | 0.0191 ($m^*=2$) | 0.0111 ($m^*=3$) | 0.0070 ($m^*=3$) |

Recall that $m^*$ represents the number of clusters which form. Here $\mu=0.2$ and $\tau=0.2$.

$\eta \to [r\tau(\sigma-1)/2]^m/(m!)$, and $g_m/\rho \to \mu\eta(2\sigma-1)/\sigma(\sigma-1)$. Here the largest growth rate will be for $m^*=1$. One cluster will form as the workers will take full advantage of the economies of scale; transportation costs are negligible and therefore there is no advantage to forming other manufacturing locations across the economic landscape.

These effects can be seen in Table 1 and in Fig. 3 (in the figure $\sigma=4$, $\tau=0.2$, and $\mu=0.2$).

As the value of $r$ or $\tau$ increases, the number of clusters which form increases.[11] For example, when $r=2$ and $\tau=0.6$, three economic clusters form. Note that in Table 1 the entries with the same value of $r\tau$ have the same value of the growth rate. For the values of $\sigma$ and $\mu$ we have chosen, the condition given earlier $\mu>(\sigma-1)/\sigma$ is *not* satisfied. In fact we can see from Fig. 3 that, for a given value of $m$, the growth rate becomes negative once the radius is sufficiently large.

In Table 2 we show the effect on the growth rate of varying $\sigma$, the elasticity of substitution. As $\sigma$ increases, there are less "economies of scale," and gains from a variety of products are lower. Thus, workers will not concentrate as much and the number of clusters which will form increases. For example, when $r=2$ and $\sigma=2$, only one economic cluster forms, while for $r=2$ and $\sigma=6$, two economic clusters form.

Finally, in Table 3 we show how the value of $\mu$, the fraction of workers in the economy, affects the number of clusters which evolve. A higher $\mu$ implies that more of the utility of the economy comes from manufacturing goods as well as that there are more workers in the economy relative to farmers. Thus, workers tend to agglomerate more as $\mu$ rises and consequently the number of clusters that emerge drops. To illustrate this last point with an extreme example, consider the case when all the laborers are workers. They are all mobile and will agglomerate into one cluster reducing actual transport costs to zero, since there are no immobile farmers who require goods in other geographical locations.

In this section, we have demonstrated how we can analytically predict the number of clusters that form given the parameters in the economy. We have rigorously obtained, without the use of simulations, the "intuitive" results stated elsewhere (Krugman, 1991, 1994): increasing the total transportation costs and "economies of scale" increases the number of clusters that form, while increasing the fraction of workers in the economy causes the number of clusters to drop.

We now proceed to examine the more realistic two-dimensional landscape.

## 4. The two-dimensional model

The analysis of the economy in one dimension is useful for it allows us to understand the underlying economic principles with few mathematical complications. However, the two-dimensional landscape is more realistic and it connects more readily with empirical observations. The economics of the two-dimensional model is identical to the one-dimensional model and we will obtain analogous results for the growth rate. The important difference is in the geometry of the clusters that form and it will be discussed in the next section.

We consider an economy located on the surface of a sphere of radius $r$. This is the natural generalization of the one-dimensional circle. Furthermore, a sphere is a convenient geometry to work with. There are no boundaries, such as those that occur on a finite two-dimensional plane, which would require additional mathematical constraints on all variables.[12] The workers and farmers are distributed along the surface of the sphere. The density of workers is represented by $\lambda(\phi, \theta, t)$ and similarly all the one-dimensional variables become a function of the two spherical angles. Here we use the usual spherical coordinates: $\phi$ is the azimuthal angle and varies between 0 and $2\pi$ and $\theta$ is the polar

[11] Tabuchi et al., (2005) have a different model where the number of cities first decreases and then increases as transportation costs increase.

[12] An infinite plane is not suitable either. The requirement that the number of laborers be finite would imply a zero density of workers and farmers.

Table 3
The growth rates $g_{m^*}/\rho$ at the dominant frequency $m^*$ for various $r$ and $\mu$

| | $\mu$=0.2 | $\mu$=0.4 | $\mu$=0.6 |
|---|---|---|---|
| $r$=1 | 0.0128 ($m^*$=1) | 0.0462 ($m^*$=1) | 0.0782 ($m^*$=1) |
| $r$=2 | 0.0117 (m*=2) | 0.0531 ($m^*$=1) | 0.1132 ($m^*$=1) |
| $r$=3 | 0.0133 ($m^*$=2) | 0.0440 ($m^*$=2) | 0.1216 ($m^*$=1) |
| $r$=4 | 0.0111 ($m^*$=3) | 0.0522 ($m^*$=2) | 0.1163 ($m^*$=1) |

Recall that $m^*$ represents the number of clusters which form. Here $\sigma$=4 and $\tau$=0.2.

angle which varies between 0 and $\pi$. The equations for the two-dimensional economy are a straightforward generalization of the one-dimensional Eqs. (12)–(17). Thus we write:

$$Y(\phi,\theta,t) = \hat{\alpha}[(1-\mu)\psi(\phi,\theta)+\mu\lambda(\phi,\theta,t)w(\phi,\theta,t)], \tag{35}$$

$$T(\phi,\theta,t) = \left[\hat{\beta}\int_0^{2\pi}\int_0^{\pi}\lambda(\phi',\theta',t)w(\phi',\theta',t)^{1-\sigma}e^{-\tau r(\sigma-1)[1-\cos\gamma]}r^2 d\Omega'\right]^{\frac{1}{1-\sigma}}, \tag{36}$$

$$w(\phi,\theta,t) = \left[\hat{\delta}\int_0^{2\pi}\int_0^{\pi}Y(\phi',\theta',t)T(\phi',\theta',t)^{\sigma-1}e^{-2\tau r(\sigma-1)[1-\cos\gamma]}r^2 d\Omega'\right]^{\frac{1}{\sigma}}, \tag{37}$$

$$\omega(\phi,\theta,t) = w(\phi,\theta,t)T(\phi,\theta,t)^{-\mu}, \tag{38}$$

$$\bar{\omega}(t) = \int_0^{2\pi}\int_0^{\pi}\lambda(\phi,\theta,t)\omega(\phi,\theta,t)r^2 d\Omega, \tag{39}$$

$$\frac{\partial\lambda(\phi,\theta,t)}{\partial t} = \rho[\omega(\phi,\theta,t)-\bar{\omega}(t)]\lambda(\phi,\theta,t), \tag{40}$$

where $\hat{\alpha}=4\pi r^2$, $\hat{\beta}=\frac{1}{\hat{\Gamma}_0(\tau r(\sigma-1))}$, $\hat{\delta}=\frac{1}{4\pi r^2\ \hat{\Gamma}_0(\tau r(\sigma-1))}$, $\cos\gamma=\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos(\phi-\phi')$, and $d\Omega'=\sin\theta'\,d\theta'\,d\phi'$ ($d\Omega$ is the solid angle in spherical coordinates). The hats over the quantities $\hat{\alpha}$, $\hat{\beta}$, and $\hat{\delta}$ are placed to distinguish them from the corresponding one-dimensional terms. We will use this convention with other quantities as well, e.g. the function $\hat{\Gamma}_0$ which is defined in the Appendix, Section A.5.

The equilibrium is: $\lambda_0(\phi,\theta)=\frac{1}{4\pi r^2}$, $\psi_0(\phi,\theta)=\frac{1}{4\pi r^2}$, $Y_0(\phi,\theta)=1$, $T_0(\phi,\theta)=1$, $w_0(\phi,\theta)=1$, and $\omega_0(\phi,\theta)$=1. This represents a uniform distribution of workers and farmers around the surface of the sphere.

The perturbed or first order equations are:

$$Y_1(\phi,\theta,t) = \hat{\alpha}\mu\left[\frac{w_1(\phi,\theta,t)}{4\pi r^2}+\lambda_1(\phi,\theta,t)\right], \tag{41}$$

$$T_1(\phi,\theta,t) = \frac{\hat{\beta}}{1-\sigma}\int_0^{2\pi}\int_0^{\pi}\left[\lambda_1(\phi',\theta',t)+\frac{(1-\sigma)}{4\pi r^2}w_1(\phi',\theta',t)\right]e^{-\tau r(\sigma-1)[1-\cos\gamma]}r^2 d\Omega', \tag{42}$$

$$w_1(\phi,\theta,t) = \frac{\hat{\delta}}{\sigma}\int_0^{2\pi}\int_0^{\pi}[Y_1(\phi',\theta',t)+(\sigma-1)T_1(\phi',\theta',t)]e^{-\tau r(\sigma-1)[1-\cos\gamma]}]r^2 d\Omega', \tag{43}$$

$$\omega_1(\phi,\theta,t) = w_1(\phi,\theta,t)-\mu T_1(\phi,\theta,t), \tag{44}$$

$$\bar{\omega}_1(t) = \int_0^{2\pi}\int_0^{\pi}\left[\lambda_1(\phi,\theta,t)+\frac{\omega_1(\phi,\theta,t)}{4\pi r^2}\right]r^2 d\Omega, \tag{45}$$

$$\frac{\partial\lambda_1(\phi,\theta,t)}{\partial t} = \frac{\rho}{4\pi r^2}[\omega_1(\phi,\theta,t)-\bar{\omega}_1(t)]. \tag{46}$$

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As in the one-dimensional model, we can perturb the density of workers along the surface of the sphere. Whatever the perturbation, it can be represented as a Laplace series (Arfken, 1970). This is a two-dimensional analog of the Fourier series used in the one-dimensional model (see discussion after Eq. (25)). Explicitly, we write the perturbation as

$$\lambda_1(\phi,\theta,t) = \sum_{n=1}^{m=\infty}\sum_{m=-n}^{m=n}\lambda_{1n}^m(t)Y_n^m(\phi,\theta), \tag{47}$$

where the $n=0$ perturbation has been dropped, since we consider only perturbations that conserve the number of workers in the economy. The functions $Y_n^m(\phi,\theta)$ are the spherical harmonics. The reader is cautioned to differentiate between this $Y_n^m(\phi,\theta)$ representing the spherical harmonics and the $Y(\phi,\theta,t)$ representing the income of various locations. The former will always be accompanied by the superscript $m$ and subscript $n$.

As before, the growth of different harmonics will be independent. Also, the harmonics with the largest growth rate will dominate the spatial structure and will determine the number of clusters that will form in the economy.

The economy will evolve according to the differential equation (46). Solving the perturbed equations as before (cf. Eq. (27)) we find that

$$\frac{\partial\lambda_{1n}^m(t)}{\partial t} = \hat{g}_n\lambda_{1n}^m(t). \tag{48}$$

where $\hat{g}_n$ is the growth rate of the harmonics of given $n$. The value of $\hat{g}_n$ is derived by solving the system of perturbed equations (41)–(44). Solving this system of equations we obtain:

$$\hat{g}_n = \rho\left[\frac{(1-\mu\hat{\eta})(\hat{\eta}\mu-\hat{\eta}^2)}{\sigma-\mu\hat{\eta}-(\sigma-1)\hat{\eta}^2} - \frac{\mu\hat{\eta}}{1-\sigma}\right], \tag{49}$$

where

$$\hat{\eta} = \frac{\hat{\Gamma}_n(\tau r(\sigma-1))}{\hat{\Gamma}_0(\tau r(\sigma-1))}, \tag{50}$$

with $\hat{\Gamma}n$ defined in the Appendix, Section A.5.

Eqs. (49) and (50) are identical to Eqs. (28) and (29) for the one-dimensional case, with the exception that $\eta$ is replaced by $\hat{\eta}$. The mathematical behavior of $\hat{\eta}$ is very similar to that of $\eta$. Thus the qualitative results that we obtained

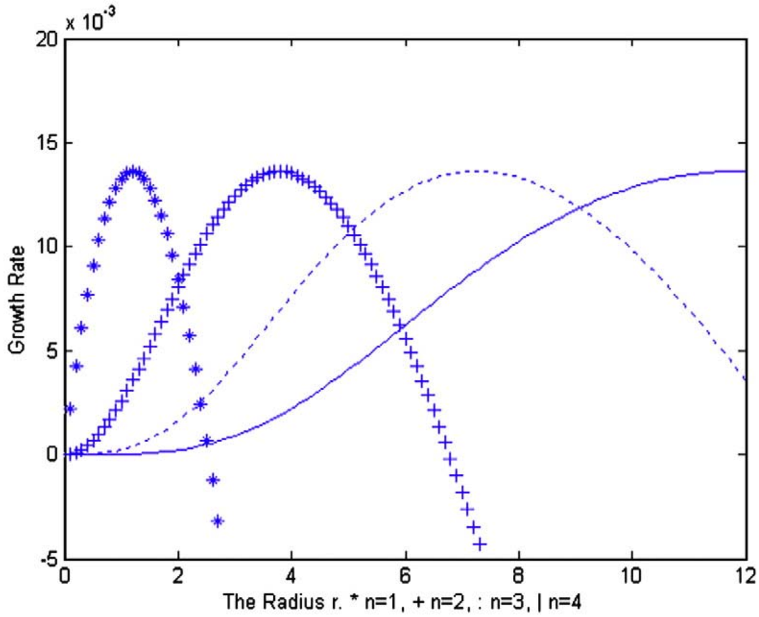


Fig. 4. The two-dimensional growth rate versus the radius $r$ for various $n$ ($\sigma=0.4$, $\tau=0.2$, and $\mu=0.2$).

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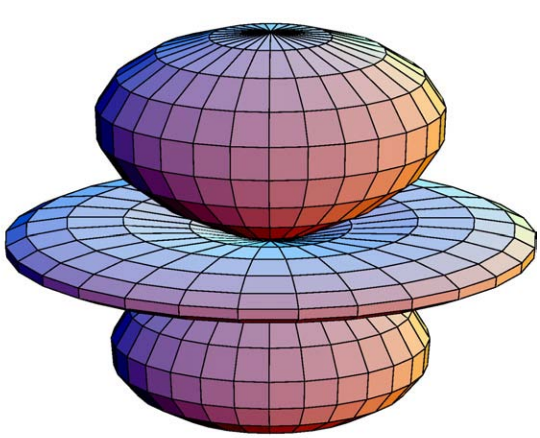

Fig. 5. Two-dimensional spatial structure for $n^*=2$: spherical harmonic $Y_2^0(\phi, \theta)$.

for the growth rate in the one-dimensional case (i.e. the variation of the growth rate with $r\tau$, $\sigma$, and $\mu$) apply here as well. The quantitative difference between the one and two-dimensional growth rates is also small as can be seen from Fig. 4 (here we use the same parameters as in Fig. 3: $\sigma=4$, $\tau=0.2$, $\mu=0.2$).

The interesting feature of the two-dimensional solution is that although the harmonics are determined by two indices, $n$ and $m$, the growth rate $\hat{g}_n$ depends only on the index $n$. This follows from the rotational invariance of Eqs. (41)–(44).

## 5. Cluster growth in two-dimensions

In this section, we do not pursue an in-depth analysis of the growth factor, since the one-dimensional and two-dimensional qualitative results are the same. In the two-dimensional model, there will be a certain $n=n^*$ with the highest growth rate. As in the one-dimensional case, the number of clusters that will form depends on $n^*$.

Let us consider a specific example. Suppose $n^*=2$ is the dominant harmonic for a given set of parameters. Thus, all of the $n^*=2$ harmonics will grow at the same rate. In Figs. 5 and 6 we show representative structures that could form.[13] Note that the amplitudes of the harmonics are arbitrary as long as they are small compared to the average density of workers $\lambda_0(\phi,\theta)=\frac{1}{4\pi r^2}$ (so that overall perturbation is small i.e. $|\lambda_1(\phi, \theta, t)|=|\lambda_0(\phi, \theta)|$). This can be seen from Eq. (48); any constant factor multiplying the harmonic will cancel out on both sides of the equation. In other words, this analysis provides the number of clusters with their relative (not absolute) amplitudes.

Our linear theory cannot say which one of these structures will dominate and it may well be that a combination of the different $n^*$ structures grows together. However, this feature suggests a mechanism for the formation of the complex structures we observe in the real world. Unfortunately with our model for the economy we are unable to investigate this idea in more detail. Nevertheless, we see how the two-dimensional nature of the economy is of primary importance. The two-dimensional model allows us to explicitly visualize the types of configurations that cities may assume in the real world. In the following section we suggest ways to pursue this question further.

## 6. Edge Cities, saturation, and further thoughts

The models (both in one and two dimensions) outlined above show how an ordered agglomeration of workers may emerge from an initially uniform economy. This regular spacing is similar to that first observed in Germany by Christaller, although it certainly is not the case that cities everywhere are distributed uniformly. We now suggest possible extensions of the model for further research. Several recent studies of cluster evolution (Chincarini, 1995; Henderson and Mitra, 1995; Krugman, 1995; Fujita et al., 1999) have attempted to model the formation of newly

[13] There are numerous online resources for visualizing the spherical harmonics. See, for example, http://web.uniovi.es/qcg/harmonics/harmonics.html; http://www.bpreid.com/applets/poasDemo.html; and http://www.vis.uni-stuttgart.de/~kraus/LiveGraphics3D/java script/SphericalHarmonics.html.

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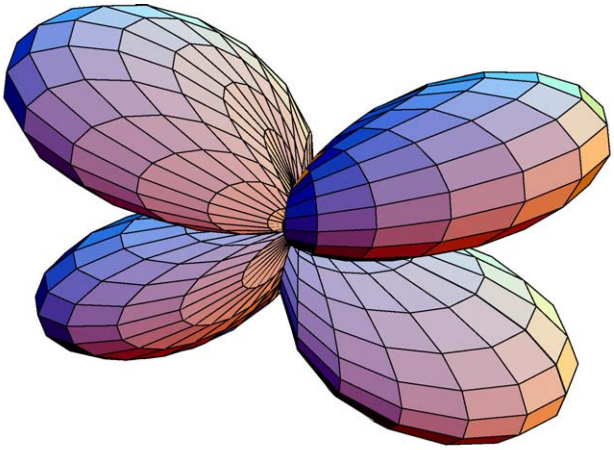

Fig. 6. Two-dimensional spatial structure for $n^*=2$: spherical harmonics $[Y_2^1(\phi, \theta)+Y_2^{-1}(\phi, \theta)]/2$.

emerged urban centers called Edge Cities. These Edge Cities, so termed because they form around major metropolitan cities, are centers that bring together jobs, market places, and residential areas. Several examples of such Edge Cities are: (i) the area around Route 128 and the Massachusetts Turnpike in the Boston region; (ii) the Schaumburg area west of O'Hare Airport near Chicago; (iii) the Perimeter Center area at the northern tip of Atlanta's Beltway; (iv) Irvine, south of Los Angeles; and (v) Emeryville, near Oakland in California.

While our model of city formation is based upon basic centripetal and centrifugal forces, such as localized industry spillovers, Edge Cities form primarily because residents and firms move out of a central city towards the suburbs or fringes of the metropolitan area.[14] While in our one-dimensional model all the cities which form are evenly spaced around a circle, our two-dimensional model allows for more complex and realistic geometries. Thus, we may envision a central city growing on the pole of a sphere with other cities growing around it. This is precisely the phenomenon illustrated in the previous section and allows for the possibility of Edge Cities. Although our model is not a model of Edge Cities, we can infer from our results that a realistic geometry is an important component of any reasonable model attempting to depict city formation.

In addition to the importance of two-dimensional models in explaining city formation, some basic building blocks must be added to the core-periphery style models. These models assume that wage differentials drive the evolution of the economy. A more complete description of the economy would be obtained by adding other factors that cause people to move. In Edge Cities, for example, workers and firms move out of a central city and towards the suburbs of the metropolitan area primarily because of increasing land rents. To account for the effect of land rents, some function which enters negatively into the residents' and firms' utility functions must be introduced. Connected with this is the issue of population growth. One reason for increases in land rents is population growth. We have explicitly assumed that the population of our economy stays constant. This assumption may be relaxed in conjunction with an introduction of land rents into the economy.

A further important reason to modify the behavior of the economy is to allow for a saturation point i.e. the establishment of a new equilibrium. The economy we have presented never reaches a new equilibrium per se. This is unsatisfactory. A possibility is an equation of motion of the following form: $\frac{\partial \lambda(\phi,\theta,t)}{\partial t} = \rho[\omega(\phi,\theta,t) - \overline{\omega}(t)][\lambda(\phi,\theta,t) - z(\phi,\theta)\lambda^2(\phi,\theta)]$. The quadratic term in $\lambda$, which could be a proxy for land rents, would allow for a saturation of the economy at a distribution of workers $\lambda(\phi, \theta)=1/z(\phi, \theta)$.[15]

In the model we have presented we study the how the economy evolves after a perturbation in the worker distribution. It is this small motion of workers that triggers the clustering of economic activity. In reality many local perturbations to the economy are not small. This is most vividly illustrated by the role developers play in the formation

[14] Some of the underlying economics of our model is supported by empirical work. See Ellison and Glaeser (1997), Ciccone and Hall (1996), Ioannides and Overman (2004), and Roos (2005). Less empirical work has been done examining the phenomena of Edge Cities, although the processes of formation seem to be quite different.

[15] Other approaches to the establishment of a new equilibrium are discussed in Quah (2002) and Ioannides (2006).

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of Edge Cities. Although many Edge Cities do evolve from an existing infrastructure, there are certain Edge Cities, the so-called "Greenfields" which form "at the intersection of several thousand acres of farmland and one developer's monumental ego" (Garreau, 1991). In such cases the developer is much more of a shock to the economy than a small perturbation. The role of developers should also be taken into account in a more complete model of the economy.[16]

We believe that the above extensions to the economy should be explored in a two-dimensional geometry. As we have shown, even for a simple economic model, there is a rich variety of city structures which form in the two-dimensional landscape. It would be interesting to explore which types of city structures evolve in two dimensions in the context of the more sophisticated economic models we have suggested and to test these findings empirically.

## 7. Conclusions

We have studied the dynamics of cluster formation in a spatial economy derived from microeconomic foundations. We use perturbation techniques to obtain analytic solutions for the temporal evolution of economic clusters. Our analytic approach permits us to relate the number of clusters formed in the economy to the underlying economic factors in a straightforward manner without the use of simulations. We have shown how we rigorously obtain the "intuitive" results found in the literature for the one-dimensional economy.

We have illustrated the strength of our formalism by presenting the heretofore unsolved two-dimensional model of city formation. We have shown how new structural features arise in two dimensions that could potentially fit the actual formation of economic clusters more accurately. With our two-dimensional solution we have established an important connection between empirical and theoretical work in the study of city evolution.

Our paper has also outlined some of the problems with current approaches to the evolution of economic clusters.[17] The main contribution of this paper is to provide an analytical model in one- and two dimensions for the core-periphery framework of Krugman. Towards this end, we have also introduced tools that may be of use in other areas, such as a rotationally invariant transport cost function. We hope that the analytic method we have presented in this paper will provide an important framework, part of "the new approach", within which to build more sophisticated and realistic models of city formation.

## Appendix A. Mathematical derivations

### *A.1. Deriving zero order and first order equations*

One may use perturbation theory to examine the evolution of a non-linear system from equilibrium by considering small perturbation of the equilibrium. Taking Eqs. (12)–(17), one can express each variable in terms of an equilibrium value (zeroth order) and a perturbed part (first order), which is assumed to be small compared to the equilibrium. Higher order terms are neglected. This restricts the analysis to examining the initial (linear) evolution of the system away from equilibrium. Thus:

$$Y(\phi,t) = Y_0(\phi) + Y_1(\phi,t), \tag{51}$$

$$T(\phi,t) = T_0(\phi) + T_1(\phi,t), \tag{52}$$

$$w(\phi,t) = w_0(\phi) + w_1(\phi,t), \tag{53}$$

$$\omega(\phi,t) = \omega_0(\phi) + \omega_1(\phi,t), \tag{54}$$

$$\overline{\omega}(t) = \overline{\omega}_0 + \overline{\omega}_1(t), \tag{55}$$

$$\lambda(\phi,t) = \lambda_0(\phi) + \lambda_1(\phi,t), \tag{56}$$

[16] The work of Quah (2002) has made some progress towards this goal by adding forward-looking agents into his model.
[17] For a more extensive criticism of these type of core-periphery models, see Scott (2004) and Neary (2001).

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One then substitutes these expressions into Eqs. (12)–(17) collecting zero and first order terms. This yields two sets of equations: one for the equilibrium terms and one for the perturbed terms. The zeroth order equations yield the values of each variable at equilibrium (see Section 2.2). The first order equations are Eqs. (18)–(23) in the text.

### A.2. Explicitly demonstrating that different frequencies are independent

We show the independence of different frequencies for one particular equation that we use in the paper. A similar argument can be applied to the remaining equations. This proof below is carried out for the one-dimensional model, but a similar argument holds for the two-dimensional equations. Consider Eq. (21):

$$\omega_1(\phi,t) = w_1(\phi,t) - \mu T_1(\phi,t). \tag{57}$$

Each perturbation may be represented as a Fourier series (recall that we exclude the $m=0$ terms in our sums)

$$\omega_1(\phi,t) = \sum_{m=-\infty}^{m=\infty} \omega_{1m}(t)e^{im\phi}, \tag{58}$$

$$w_1(\phi,t) = \sum_{m=-\infty}^{m=\infty} w_{1m}(t)e^{im\phi}, \tag{59}$$

$$T_1(\phi,t) = \sum_{m=-\infty}^{m=\infty} T_{1m}(t)e^{im\phi}, \tag{60}$$

where $w_{1m}(t)$, $w_{1m}(t)$, and $T_{1m}(t)$ are time-dependent coefficients. Substituting Eqs. (58)–(60) into Eq. (57):

$$\sum_{m=-\infty}^{m=\infty} \omega_{1m}(t)e^{im\phi} = \sum_{m=-\infty}^{m=\infty} w_{1m}(t)e^{im\phi} - \mu \sum_{m=-\infty}^{m=\infty} T_{1m}(t)e^{im\phi}. \tag{61}$$

Multiplying the above equation by $e^{-il\phi}$ and integrating over the circumference of the circle, we have:

$$\sum_{m=-\infty}^{m=\infty} \int_0^{2\pi} \omega_{1m}(t)e^{i(m-l)\phi} r d\phi = \sum_{m=-\infty}^{m=\infty} \int_0^{2\pi} w_{1m}(t)e^{i(m-l)\phi} r d\phi - \mu \sum_{m=-\infty}^{m=\infty} \int_0^{2\pi} T_{1m}(t)e^{i(m-l)\phi} r d\phi. \tag{62}$$

Using the orthogonality relation of the Fourier harmonics,

$$\int_0^{2\pi} e^{i(m-l)\phi} d\phi = 2\pi\delta_{lm}, \tag{63}$$

where $\delta_{lm}=1$ for $l=m$ and is zero for $l\neq m$, Eq. (62) becomes:

$$\omega_{1l}(t) = w_{1l}(t) - \mu T_{1l}(t). \tag{64}$$

We see that the original equation has become an equation involving only the coefficients of one particular frequency in the general perturbation. This procedure can be applied to the whole system of Eqs. (18)–(23). The result will be a series of linear equations involving the coefficient of a particular Fourier harmonic e.g. Eq. (26). From these equations, the growth rate, $g_m$, as given by Eqs. (27) and (28) may be obtained.

### A.3. Proof that $\bar{\omega}_1=0$

From Eq. (22) we have

$$\bar{\omega}_1(t) = \int_0^{2\pi} \left[\lambda_1(\phi,t) + \frac{\omega_1(\phi,t)}{2\pi r}\right] r d\phi. \tag{65}$$

Expanding all variables in a Fourier series (see Eq. (24))

$$\bar{\omega}_1(t) = \sum_{m=-\infty}^{m=\infty} \left[\lambda_{1m}(t) + \frac{\omega_{1m}(t)}{2\pi r}\right] \int_0^{2\pi} e^{im\phi} r d\phi = 0, \tag{66}$$

since $m \neq 0$.

### A.4. The function $\Gamma_m(\tau r(\sigma-1))$

In the course of solving Eqs. (18)–(23) the following integral recurs

$$K_m(\phi) \equiv \int_0^{2\pi} e^{-\tau r(\sigma-1)(1-\cos(\phi-\phi'))} e^{im\phi'} d\phi'. \tag{67}$$

Specifically, the integral occurs when the perturbed variables in Eqs. (19) and (20) are expanded in a Fourier series. The integral in Eq. (67) may be evaluated as follows. First we write

$$K_m(\phi) = e^{-z} \int_0^{2\pi} e^{z\cos(\phi-\phi')} e^{im\phi'} d\phi', \tag{68}$$

where $z = \tau r(\sigma-1)$.

Now, we use the identity (Gradshtein and Ryzhik, 1994)

$$\int_0^{2\pi} e^{z\cos(\phi-\phi')} e^{im\phi'} d\phi' = 2\pi I_m(z) e^{im\phi}. \tag{69}$$

Here $I_m$ $(z)$ is the modified Bessel function of the second kind (Arfken, 1970). Therefore Eq. (68) becomes

$$K_m(\phi) = 2\pi e^{-z} I_m(z) e^{im\phi}. \tag{70}$$

It is convenient to introduce the function $\Gamma_m(z) \equiv e^{-z} I_m(z)$. Thus

$$K_m(\phi) = 2\pi \Gamma_m(z) e^{im\phi}. \tag{71}$$

It is through Eq. (71) that quantity $\eta = \Gamma_m(z)/\Gamma_0(z)$ appears in the growth rate $g_m$. The function $\Gamma_0$ $(z) = \Gamma_{m=0}$ $(z)$ enters through the normalization constants $\beta$ and $\delta$ (see Eqs. (13) and (14)). These are also obtained using Eq. (71).

### A.5. The function $\hat{\Gamma}_m(\tau r(\sigma-1))$

The integral which occurs in the solution of the perturbed two-dimensional equations is: (cf. Section A.4 of this Appendix)

$$\hat{K}(\phi,\theta,t) = \int_0^{2\pi} \int_0^{\pi} e^{-z[1-\cos\gamma]} Y_n^m(\phi',\theta') d\Omega' \tag{72}$$

$$= e^{-z} \int_0^{2\pi} \int_0^{\pi} e^{z\cos\gamma} Y_n^m(\phi',\theta') d\Omega'. \tag{73}$$

To evaluate this integral, we can expand $e^{z\cos\gamma}$ as a Legendre series (Arfken, 1970):

$$e^{z\cos\gamma} = \sum_{k=0}^{k=\infty} a_k P_k(\cos\gamma). \tag{74}$$

Thus, Eq. (73) becomes:

$$\hat{K}(\phi,\theta,t) = e^{-z}\sum_{k=0}^{k=\infty} a_k \int_0^{2\pi}\int_0^{\pi} P_k(\cos\gamma)Y_n^m(\phi',\theta')d\Omega' \tag{75}$$

$$= e^{-z}\sum_{k=0}^{k=\infty} a_k \frac{4\pi}{2k+1} Y_n^m(\phi,\theta)\delta_{nk} \tag{76}$$

$$= e^{-z}\frac{4\pi a_n}{2n+1} Y_n^m(\phi,\theta) \tag{77}$$

$$= 4\pi e^{-z} i_n(z) Y_n^m(\phi,\theta) \tag{78}$$

$$= 4\pi \hat{\Gamma}_n(z) Y_n^m(\phi,\theta), \tag{79}$$

where $z=\tau r(\sigma-1)$, $\hat{\Gamma}_n(z)=e^{-z}i_n(z)$, and $i_n(z)$ is a spherical modified Bessel function of the second kind (Arfken, 1970).

In Eq. (77) we have used the result $a_n=(2n+1)i_n(z)$. The derivation is given below in Section A.6.

### A.6. Derivation of $a_n$

We defined $a_n$ through Eq. (74). Using the orthogonality of Legendre polynomials (Arfken, 1970) and substituting $x=\cos\gamma$, we have:

$$a_n = \frac{2n+1}{2}\int_{-1}^{1} P_n(x)e^{zx}dx. \tag{80}$$

It is also true that:

$$j_n(q) = \frac{1}{2i^n}\int_{-1}^{1} P_n(x)e^{iqx}dx, \tag{81}$$

where $j_n(q)$ is the (normal) spherical Bessel function (Arfken, 1970).

Making the substitution, $q=-iz$, we obtain:

$$j_n(-iz) = \frac{1}{2i^n}\int_{-1}^{1} P_n(x)e^{zx}dx. \tag{82}$$

Therefore:

$$a_n = \frac{2n+1}{2} 2i^n j_n(-iz) \tag{83}$$

$$= (2n+1)i^n(-1)^n j_n(iz) \tag{84}$$

$$= (2n+1)i^{-n} j_n(iz) \tag{85}$$

$$= (2n+1)i_n(z). \tag{86}$$

In the last step of the derivation we have used the identity $i_n(z)=i^{-n}j_n(iz)$.

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