Computer Simulation of the Alonso Household Location Model in the Microeconomics Course

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Abstract: Computer simulation of the Alonso household location model can enrich the intermediate microeconomics course. The model includes decisions on location, land space, and other goods and is a valuable complement to the usual textbook model of household consumption. It has three decision variables, one of which is a "bad," and one good's price is a nonlinear function of another decision variable. These instructive complications are easily within the grasp of students. The simulation illustrates algebraic utility functions that are important in later courses. The author describes ways to simulate various versions of the model, from relatively simple to advanced, including a version that incorporates time-allocation decisions, thus giving the instructor flexibility in teaching students of varying ability.

Key words: Alonso model, computer simulation, teaching microeconomics
JEL codes: A22, D11, J22

Intermediate microeconomics is a vital course in the economics major and also an important course in professional programs in city planning, public policy, and business. Teaching it is often frustrating because textbooks use simple examples that are too unrealistic for students, especially mature students in professional programs. Instructors can improve the course by demonstrating computer simulations of more realistic theoretical models. Simulations help students see strengths and weaknesses of models more clearly, and they can be more like the real world than typical graphical expositions because they allow more decision variables. Thus, they supplement the standard graphical techniques. They also allow the instructor to show more numerical examples in class and allow students to do more problems out of class.

In this article, I describe one especially useful simulation model in this regard, a model of a household's choice of residential location, land space, and other

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consumer goods. It is based on a theoretical model called the Alonso model, after William Alonso, who developed its characteristics extensively (Alonso 1964). Alonso developed it for the field of urban economics, where it is now standard fare. I argue that it is also very useful in teaching microeconomics, quite independent of its role in urban economics, although some of the article may be useful to teachers of urban economics as well.

The Alonso model supplements, but does not replace, the typical household model that has two goods and constant prices. After developing the standard model, the teacher can use the Alonso model to introduce realism by adding a "bad" in the preference function and by altering the budget constraint to make one good's price endogenous rather than an exogenous constant. This can be done easily without requiring more prior experience in economics than the typical intermediate student has, or more complicated mathematics than he or she can learn in the course. The model's features anticipate work later in the major, such as in labor economics, urban economics, transportation economics and planning, real estate, and public finance. These are important advantages for both undergraduates and professional students.

I describe several versions of the model, offering instructors considerable choice and allowing them to tailor an approach to their own students. I describe simulations with both Cobb-Douglas and CES (constant elasticity of substitution) utility functions, but some instructors will choose only one or the other. I also show how to use two advanced concepts—indirect utility functions and time allocation—but certainly a teacher can use the model fruitfully without venturing into such complications. As a minimum, the instructor can describe the model generally, explain some three-dimensional diagrams of the budget surface and indifference surfaces, and then use a spreadsheet program to do numerical simulations in class. He or she can show the spreadsheet results in ways ranging from simple tables (my own approach) to elaborate PowerPoint presentations and graphics. However, many students can easily program a spreadsheet version of the model on their own and solve appropriate homework problems. Students not already familiar with spreadsheets generally appreciate the opportunity to learn to use them. If students know a more advanced mathematical optimization program, the instructor can use it in class or have students use it in homework; this might be the case in engineering, transportation, and finance programs.

I first describe generally the advantages of using the Alonso model in the classroom. Next I describe the underlying theoretical model and then describe details of some simulation versions, especially in regard to different utility functions (two different Cobb-Douglas and two different CES functions). I suggest class or homework exercises. I then explore the bid-rent approach, which is an alternative method to find the household's optimum that has the advantage of exploiting the indirect utility function, an advantage if students are more advanced and the instructor has time to teach that concept. The bid-rent approach is used widely in urban economics, but a few instructors might want to use it in intermediate micro. Finally, I describe a combination of an Alonso model and a Becker time-allocation model, which is suitable only for fairly advanced students and requires considerable teaching time.

In the model, an urban household chooses *residential location* in addition to consuming two market goods. Location is defined as the distance, D, from employment; increasing D increases the monetary cost of commuting, C, and also reduces utility because it takes time away from leisure and other desirable activities, causes boredom or tension, and increases risk of delay and accident. Thus distance is a bad. The two market goods are L, land space occupied (size of the residential lot), and Z, a composite good equal to expenditure on everything except land space and commuting costs. The price of $Z = 1$, so Z is a *numeraire*,[1] and the price does not vary with location. Income, Y, is exogenous in the most common version of the model, but the time allocation extension I describe later makes income and leisure endogenous.

The household maximizes utility by choosing L, Z, and D within the budget constraint that expenditures on L and Z, plus C, cannot exceed Y. The price of L is not constant but decreases as distance increases. This is realistic if household members commute to a city center or other employment center, as urban economists usually assume when embedding the model in a larger model of land use and land prices. In a microeconomics course, it is not necessary to invoke any specific urban economics models. Rather, the key feature is interdependence: The price of one good, land, depends on the value of the bad, D, that the household chooses.

As I show in the next section, the Alonso model allows the teacher to reinforce basic concepts such as opportunity cost, marginal cost, composite good, *numeraire*, suboptimization, local maxima vs. global maxima, and Marshallian demand functions. It adds complexity and realism because the household has three decision variables instead of two; the fact that one is a bad also adds realism and anticipates other courses where bads are prominent. The nonconstant price for land illustrates a nonlinear budget constraint, and it reinforces the distinction between average and marginal cost. The marginal cost of increasing one decision variable, D, depends on both commuting cost and land price, and this marginal cost may be positive or negative—it will be negative if the decrease in land price exceeds the increase in commuting cost. It must be negative at the household's optimum, which demonstrates the general principle that assumptions of optimal behavior impose constraints on numerical values in a model solution.

These features offer opportunities for effective teaching in preparing students for other courses, as well as in the microeconomics course. Bads are important in labor economics (work), environmental economics (pollution), and finance (risk). Nonconstant prices are important in labor economics (overtime wage rates, implicit marginal price of labor to a monopsonist), public finance (progressive tax rates, tax exemptions), and transportation economics (typical fare structures). Negative marginal costs are important in health economics (e.g., for preventive care, which is costly in itself but reduces the expenditure required for other inputs into good health).

The underlying theory has weaknesses, as in all models. It is a partial equilibrium model of the household, not a general equilibrium model of land markets (Alonso embedded his model in such a general equilibrium model). There is a single utility function for the entire household, which ignores intrahousehold conflict and bargaining. Much is left exogenous, including location of employment. However, computer simulation helps the teacher and students discuss weaknesses of theory in a more satisfying way than the usual graphic methods because results that are numbers reveal weaknesses more clearly. Students' questions and doubts about theory are no less challenging with simulations, but the discussion of them can be more productive. When a simulation turns up something "odd," students can learn more easily whether the model is odd, or they assumed odd parameters, or they do not fully understand what is odd and what is not.

The teacher can choose a simulation method from a wide variety that are readily available. Spreadsheet programs are the easiest for all concerned, and they are adequate for all the simulations described here. A teacher who has little experience in simulation and little time to prepare probably should use a spreadsheet to solve for the household's optimum by iteration, as detailed here. Iteration is easy and straightforward, and it actually has pedagogical advantages. Some teachers and students know more elaborate programs like Visual Basic, Mathematica, Mathcad, GAMS, and GAUSS and should be able easily to use one of them to simulate the model after reading the article. Because teachers' preferences and abilities vary widely, I do not discuss specifics of simulation methods.

THE THEORETICAL MODEL

The wage rate, number of wage earners and the hours they work, the location of work, and nonwage income are all exogenous, so total income, Y, is exogenous (this is relaxed in the time-allocation model discussed later).

Preferences

The utility function is

$$U = U(L, Z, D), \tag{1}$$

where: L = land space; Z = a composite good equal to monetary expenditure on all goods except land space and monetary costs of commuting; and D = distance from residence to employment.

L and Z affect U positively, D negatively. In urban economics, most users of the model prefer to simplify it by *not* having distance affect utility directly but having it determine the monetary cost of commuting and perhaps also work time and thus reduce income available for goods (Muth 1969, 21; Fujita 1989, 43; Mills and Hamilton 1994, 107, 111). They make that simplification to derive easily some standard results in the monocentric city model (which Alonso noted is hard to do when distance is in the utility function [1964, 106–09]). However, I believe that in a general microeconomics course it is important to follow Alonso's formulation by having distance (D) affect utility negatively, so students gain experience with a bad.

The variable L is *land space* only and excludes the housing *structure*—bricks, wood, roofing, and so forth—which is included in Z, on the assumption that its price, for any given size and quality of construction, does not vary with distance. However, the specification can be changed to combine land space and structure in a single variable, Housing, whose price does vary with distance (Muth 1969, ch. 2). The Alonso model's inclusion of the housing structure in Z is a simplification, and the model does not explain how big a house or apartment the household chooses. In the real world, households who choose large lots also tend to choose large structures, but the model is silent on size of structure. One can expand the model to have four decision variables—L, H, Z, and D—where now H is the size of structure, and Z is expenditure on everything except L, H, and commuting cost. This expansion requires putting H in the utility function and specifying a price or price function for H. This four-variable model is much more complicated than the three-variable one I describe here, and in my judgment, does not add enough to the three-variable model to justify the added complications, but other instructors may disagree.

The *quality* of land is identical at all distances (the teacher can refer to a von Thünen plain). This is patently unrealistic: The quality of land really depends on its physical and public environments. Students will benefit from discussing how the model is unrealistic for that reason.

Budget Constraint and Prices

The budget constraint is

$$C + P_L L + Z = Y, \tag{2}$$

where Y = income, C = roundtrip commuting costs, and both C and P_L are functions of D.

It is useful to stress to students that P_L is a rent, or user cost, not an asset price. In consumer theory, the price of land or a durable good is a rental price if the household rents, an opportunity cost of ownership if it owns. If the teacher has not yet introduced this distinction as an application of opportunity cost, here is a context to do so. Empirically, P_L declines with D, at a decreasing rate, so an exponential or quadratic function can be assumed in a simulation.

The model is a partial equilibrium one, like so many models in the intermediate micro course. The P_L function is exogenous to the household, but in the real world, it is determined endogenously in a general equilibrium. General equilibrium analysis would show that P_L cannot fall below the value of land in other uses, such as agricultural or industrial, which implicitly constrains the shape of the function that one can assume in a partial equilibrium analysis. The general equilibrium analysis of land prices is typically found only in an urban economics course. In intermediate micro, it seems sufficient to warn students of the partial equilibrium nature, to note that not all values of P_L are compatible with general equilibrium, and to discuss in class the reasonableness (in the students' own region) of any P_L function that is assumed.[2]

Commuting cost, C, includes out-of-pocket monetary costs—transit fares, fuel, parking, and so forth—for all commuters in the household. The most plausible shape has C increase with D at a decreasing rate because some elements of monetary cost are fixed and some are variable. The model incorporates many negative aspects of commuting beyond the monetary cost because D is in the utility function; when one considers both the utility and the monetary cost, it is not necessarily true that the real cost increases at a decreasing rate. The time allocation version that I discuss later has commuting time cut into income as well as leisure.

An increase in D has two effects on expenditure: It increases monetary commuting cost, but it also decreases the price of land. The *marginal cost of distance* (MCD) is the effect of a small change in D on total expenditure—the amount required to buy given quantities of L and Z, plus C. It is the partial derivative of expenditure with respect to D, holding L and Z constant as follows:

$$MCD = \partial(C + P_L L + Z)/\partial D = dC/dD + (dP_L/dD)L. \quad (3)$$

The first term is the marginal commuting cost, which is positive, and the second is the land price effect, which is negative and depends on the amount of land purchased as well as on the slope of the price function. The sum may be positive or negative.

The following thought experiment is useful: The household starts at a hypothetical location, consuming some amounts of L and Z and then considers increasing D a small amount. The move will increase the first term in equation (3)—dC/dD is positive—and decrease the second—$(dP_L/dD)L$ is negative. If the first term is larger, the household will have to buy less L and Z to remain within its budget, so the move will be undesirable on all counts because lower L and Z and increased commuting all reduce utility. If the second term is larger, then the household can buy the original amounts of L and Z at less cost, have income left over, and increase utility by spending that freed income on more L and Z. Whether it actually moves depends on how important the increased utility from L and Z is compared with the disutility of more commuting.

Alonso did not use the phrase *marginal cost of distance*, preferring "*marginal cost of spatial movement*" (Alonso 1964, 34); another possible term is the *price of distance*.

Condition for an Optimum Location

One first-order condition for an optimum (ruling out corner solutions) is that the monetary equivalent of the marginal utility of distance (MMUD) equals the marginal cost of distance (MCD):

$$(\partial U/\partial D)/(\partial U/\partial Z) = dC/dD + (dP_L/dD)L.$$

Students who know calculus can be asked to derive this condition from first-order conditions for maximum U. $MMUD$ is always negative, because $\partial U/\partial D$ is negative, so at an optimum MCD must also be negative, and, because dC/dD is positive, that means dP_L/dD must be negative. (If there is a corner solution optimum at $D = 0$, MCD could be positive.)

In most adaptations of the Alonso model in urban economics courses, D does not appear in the utility function, so $\partial U/\partial D = 0$. This is the case for the Muth model that is popular in urban economics (Muth 1969, 22–23). In that case, the above condition reduces to

$$dC/dD + (dP_L/dD)L = 0, \text{ or } dC/dD = -(dP_L/dD)L,$$

so that marginal commuting cost must equal marginal land price savings. This is the Muth Condition (see also Fujita 1989, 25). If I multiply the right-hand side by P_L/P_L and rearrange, I have

$$dC/dD = (LP_L)[-(dP_L/dD)/P_L],$$

or marginal commuting cost equals expenditure on land times the relative decrease in land price per unit distance (Muth 1969, 23). However, as I argued earlier, in intermediate micro, it is preferable to rely on Alonso's original formulation in which distance enters the utility function so that students gain experience with a bad.

USEFUL DIAGRAMS

All teachers of intermediate micro use diagrams. I show the three-dimensional indifference surface in Figure 1 and the three-dimensional budget surface for the Alonso model in Figure 2. All points on the surface in Figure 1 are equally satisfactory. The shape is the typical one for the case of two goods and one bad. The line *ab* is a conventional two-good indifference curve in the Z–L plane, at $D = 0$; lines *ce*, *fg*, and *hk* are similar indifference curves, each one holding D constant at some value. The line *ah* is an indifference curve between D and Z, for constant L, and shows that as D increases the household must have more Z to compensate.

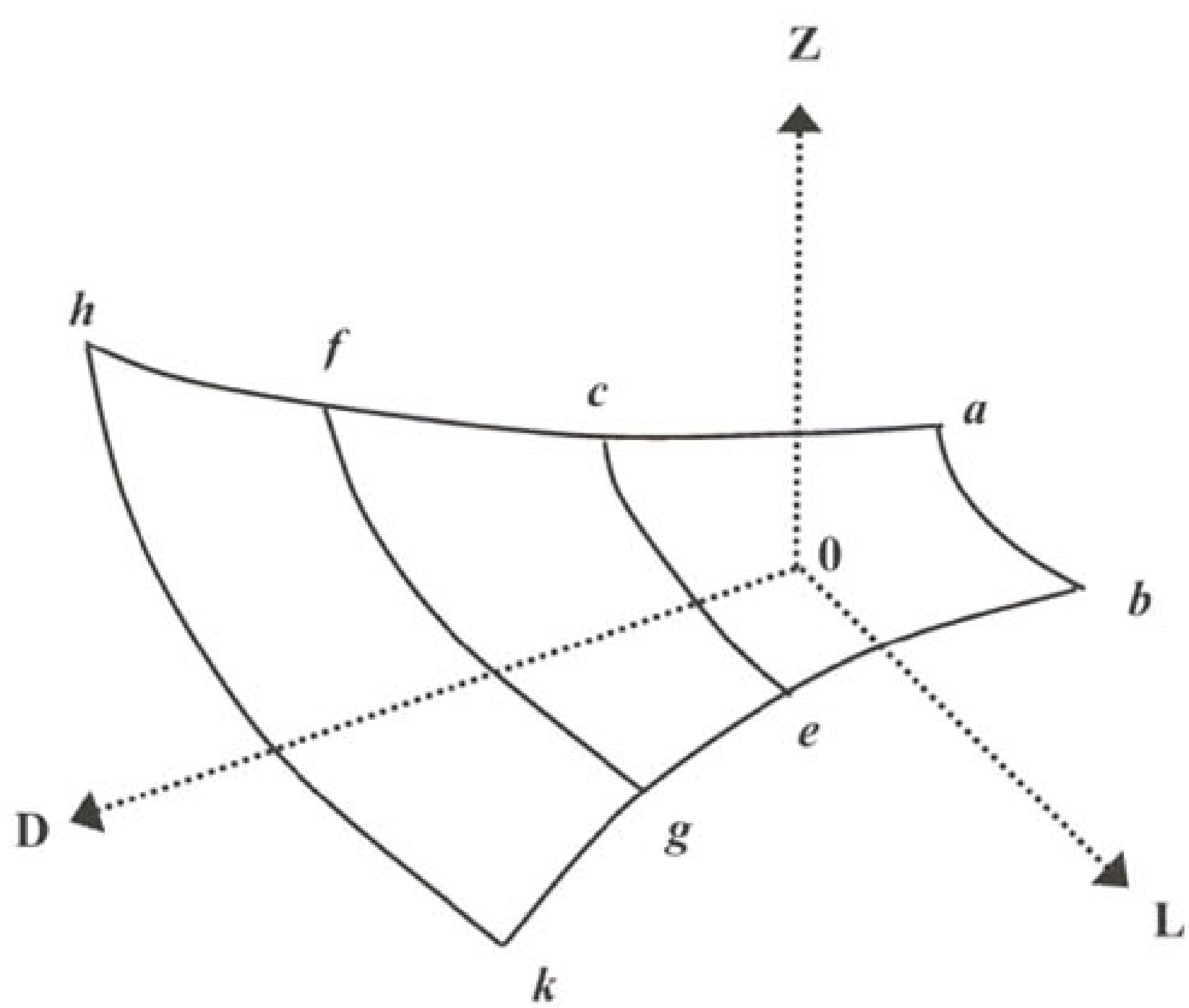

FIGURE 1. Indifference surface.

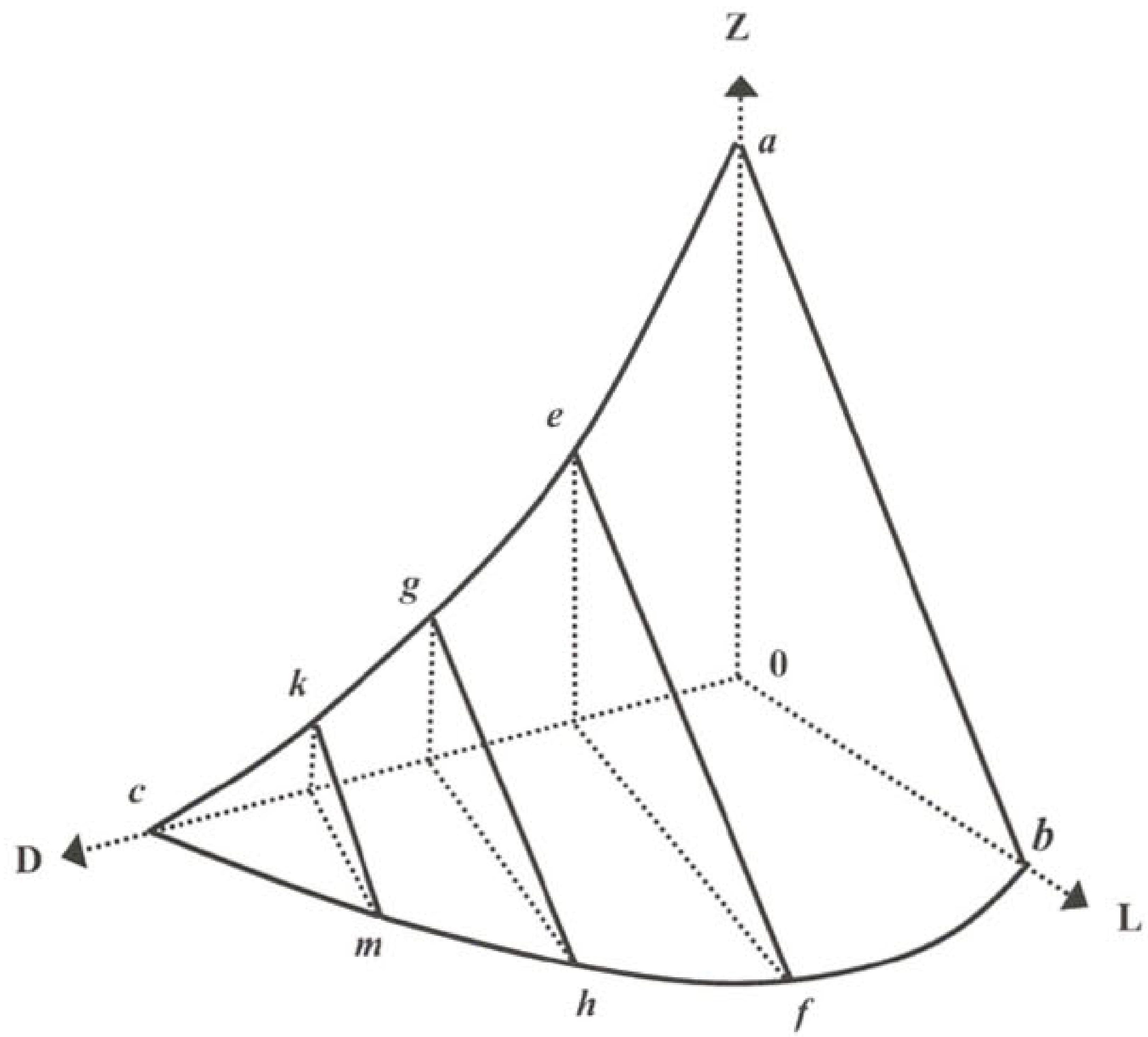

FIGURE 2. Budget surface.

The line *bk* is an indifference curve between *D* and *L*, for constant *Z*, and shows that as *D* increases there must be more *L* to compensate.

I show in Figure 2 a solid that is a budget surface, containing all the feasible combinations of *Z*, *L*, and *D*, given the household's income. The figure is similar to Alonso's Figure 5 (1964, 25) but from a different viewpoint. Some points are extreme consumption points—corners—that are limiting cases that no real household would choose: *c*, for example, where all income goes for commuting and none for *Z* or *L*; and *b*, where the household lives at its workplace and consumes only land. Lines *ab*, *ef*, *gh*, and *km* are conventional constant-price budget lines for a two-good choice between *L* and *Z*, holding *D* constant in each case. These are linear because P_L is constant at any given *D*; as *D* increases, P_L decreases and so the lines become flatter. Line *bc* (similar to Alonso's two-dimensional Figure 3, p. 23) shows the trade off between *L* and *D* when $Z = 0$; as *D* increases, the line rises at first, reflecting the reasonable assumption that the fall in P_L offsets the rise in *C* and thus allows more land to be purchased; but beyond some point, the line falls as the rise in *C* offsets the fall in P_L. Finally, line *ac* shows the trade off between *D* and *Z* when $L = 0$, and its shape assumes *C* rises at a decreasing rate with *D*.

THE COMPUTER SIMULATION MODEL AND A NUMERICAL EXAMPLE

In this section, I describe four different utility functions the instructor might use in a simulation. I go into detail for the first one, a simple modification of a Cobb-Douglas function, and present a numerical example, then discuss the other three more briefly. These are another modified Cobb-Douglas and two modified CES functions. Variables Y and Z are in monetary units per month, L in acres or hectares, D in miles or kilometers, and P_L in monetary units per acre or hectare per month.

Modified Cobb-Douglas Utility Function

This case is easy, suitable for students with little background in economics.

Utility function: $U = AL^{\alpha}Z^{\beta} - gD^{\gamma} \qquad \alpha, \beta, g > 0; \gamma > 1. \qquad (4)$

The add-on term makes it clear that D is a bad, and simulations are intuitive to students who know about marginal rates of substitution and price and income elasticities (as they will if the teacher uses this model after developing the two-good, constant-prices case). The relative sizes of α and β are important but not their sum because a monotonic transformation of U would not change the optimum. Specifying $\gamma > 1$ gives increasing marginal disutility of distance, the typical assumption. Students must choose values for g and γ that make the size of the add-on term realistic; an add-on term as high as, say, 30 percent of the first term would be questionable.

Land price function: $P_L = P_o\exp(\delta D) \qquad \delta < 0. \qquad (5)$

In this exponential function, $P_L = P_o$ if $D = 0$ and declines as D increases.

Commuting cost function: $C = \nu D^{\eta} \qquad \nu > 0; \quad 0 < \eta < 1. \qquad (6)$

Having $\eta < 1$ ensures decreasing marginal cost for the monetary cost of commuting.

Solution: The household maximizes U, subject to the budget constraint. The easiest solution method is to iterate over small increments in D, optimizing L and Z at each step by using Marshallian demand functions (shown below), then choose the value of D that gives the highest U. Denote this maximizing value of D by D^*. The value of D^* and the optimal L and Z at D^* constitute the optimum. The fact that the solution is by iteration over D does not make D a more important decision variable than L or Z; all three variables, L, Z, and D are in the utility function and the costs of all three are in the budget constraint, so all three are important. The household is certainly not driven by any simple desire to minimize commuting cost: It may be willing to pay high commuting cost to get a low land price. Simulations will show that. Iteration has two pedagogical advantages: It shows the convergence of the marginal cost of distance with the monetary equivalent of the marginal disutility of distance, and it also allows students to inspect nonoptimal solutions and understand *why* they are nonoptimal. If students are more sophisticated, however, the instructor may choose to use mathematical optimization routines that solve the problem without showing any iteration explicitly.

If the instructor does choose iteration, the procedure is as follows:

Step 1. Assume $D = 0$.
Step 2. Calculate P_L and C by equations (5) and (6).
Step 3. Calculate $M = (Y - C)$ = total expenditure on L and Z combined.
Step 4. Calculate values of L and Z that maximize U, using these Marshallian demand functions:

$$L^* = [\alpha/(\alpha + \beta)][M/P_L]. \tag{7}$$

$$Z^* = [\beta/(\alpha + \beta)][M/P_Z]. \tag{8}$$

The instructor may want to emphasize these functions, especially if he or she has already discussed Marshallian demand functions in the simpler two-good model. One could suppress P_Z because it equals one, but showing it explicitly highlights the similarity between the two functions. (Students who know calculus can be asked to derive these demand functions from first-order conditions for an optimum.)
Step 5. Choose another value of D, slightly higher, and repeat Steps 2 through 4.
Finally, scan all values of D to find the one that maximizes U.

At each increment of D, the student can see how U, L^*, and Z^* have changed from the previous step, which is useful pedagogically, especially in the early stages of learning, even if it requires doing each step slowly and stopping frequently for discussion. At each level of D, it is useful to compute the marginal cost of distance, *MCD*, and the monetary equivalent of the marginal utility of distance, *MMUD*. The *MMUD* is always negative. If $MCD < MMUD$ (that is, *MCD* is more negative than *MMUD*), then the household should increase D; if $MCD > MMUD$ (that is, if it is less negative or is positive), then the household should decrease D; if $MCD = MMUD$, the household is at an optimum (but it might be only a local optimum).

The instructor might use iteration to do simulations when first teaching the model but then, after students have gained familiarity, switch to some other optimization program that displays only the final solution. An instructor who uses a spreadsheet program can program it to display only the final solution, with judicious use of functions like MATCH and MAX in Excel.

A numerical example of this Cobb-Douglas case is $Y = 7{,}000$; $A = 1$; $\alpha = .05$; $\beta = .90$; $g = 15$; $\gamma = 1.75$; $P_O = 2400$; $\delta = -.5$; $v = 30$; $= \eta = .75$. The solution to the nearest .01 mile is: $D^* = 3.06$; $L^* = .7019$; $Z^* = 6{,}565.82$. At this solution, $P_L = 519.69$, and $C = 69.41$; $U = 2{,}572.197$; $MCD = -165.37$, and $MMUD = -165.42$, the two being equal except for trivial rounding error in D^*. The household's various expenditures are: C = commuting cost = 69.41; $P_L L$ = expenditure on land = .7019 times 519.69 = 364.77; and Z = expenditure on all else = 6,565.82. These add up to income = 7,000; the adding up exercise is useful pedagogically.

An advantage of simulation models is that they give students insights into what sorts of theoretical models and what parameter values are reasonable. Most students will gain these insights only if they simulate models a number of times, varying functional forms and parameter values, and that is true of the Alonso model. A special advantage of the model is that it is about household decisions that are relatively familiar to many students, as they usually come to the course with knowledge of their own families' choices. For example, students would suspect a solution that has a low income household choosing to live on a large lot, paying a very high land price, and having only a small proportion of total income left to consume Z. This advantage is one that simulation of some other models are less likely to have, for example, models of oligopoly interaction, labor markets, firms' finances and portfolio allocation, and public decisionmaking. The residential location choice is more familiar to most students than many other decisions by households and firms.

Another Modified Cobb-Douglas Utility Function

If an add-on term for disutility of distance is unappealing, an alternative is

$$U = AL^{\alpha}Z^{\beta}\,[c(1-hD)^{\gamma}] \qquad \alpha, \beta, c, h > 0; \qquad 0 < \gamma < 1; D < 1/h. \qquad (4A)$$

The factor in brackets is an accessibility variable and declines as D increases. It operates as a multiplier on a regular Cobb-Douglas function. It equals c when $D = 0$, and equals 0 when $D = 1/h$; obviously one must choose c and h small enough to make the specification realistic. Specifying γ to be positive but less than 1 implies increasing marginal disutility of D, and the marginal disutility of D is greater the greater are L and Z, which makes sense if commuting time comes at the expense of time the household has to enjoy L and Z.

The steps in each iteration are the same as before; the optimal values L^* and Z^* are calculated as in equations (7) and (8) in step 4 in the previous case.

CES Utility Functions

Teachers know the undesirable features of Cobb-Douglas functions: the elasticity of substitution between L and Z is always one; the price elasticities of demand for L and Z are always one. At the optimum, expenditure on land always equals the fraction $\alpha/(\alpha+\beta)$ of M, the total expenditure on L and Z combined, and expenditure on Z always equals the fraction $\beta/(\alpha+\beta)$ of M, no matter what D and P_L are. In the numerical example given earlier, expenditure on land always equals 5.3 percent (.05/.95), and expenditure on Z equals 94.7 percent (.90/.95) of M.

An alternative utility function common in empirical research is the CES function, and computer simulations of the Alonso model are one way to introduce or reinforce it in the microeconomics course, thus helping prepare students to read empirical work later. Using it is not attractive, but may be useful,if one has especially weak or nonmathematical students, for whom it is more appropriate to rely entirely on the Cobb-Douglas.

Again, one can use an additive term for disutility of distance

$$U = (aL^{\rho} + bZ^{\rho})^{1/\rho} - gD^{\gamma} \qquad a, b, g > 0; \qquad 0 \neq \rho < 1\,; \gamma > 1. \qquad (4B)$$

The first term is a standard CES form.[3] The parameter $\rho = (\sigma - 1)/\sigma$, where σ = elasticity of substitution (of course, $\sigma = 1/(1 - \rho)$, but σ is the fundamental parameter, and ρ is merely a transformation for notational convenience). If $\sigma = 1$, then $\rho = 0$, and L'Hôpital's Rule shows the function (4B) is actually the Cobb-Douglas function. The parameter ρ can be positive or negative but must be less than +1; if it is positive and increases toward +1, substitutability increases and, in the limit, indifference curves between L and Z are linear rather than convex to the origin; if it is negative and decreases toward $-\infty$, indifference curves approach the L-shaped or Leontief form. The relative sizes of a and b determine the shares of $(Y - C)$ spent on L and Z, respectively, but those shares depend on P_L, unlike in the Cobb-Douglas function.

If one solves by iteration, the iteration proceeds as before, except that in step 4 the Marshallian demand functions are

$$L^* = [a^\sigma P_L^{r-1}/(a^\sigma P_L^r + b^\sigma P_Z^r)]M, \tag{9}$$

$$Z^* = [b^\sigma P_Z^{r-1}/(a^\sigma P_L^r + b^\sigma P_Z^r)]M, \tag{10}$$

where $r = \rho/(\rho-1)$. Again it is useful to show P_Z explicitly, even though it equals one, to highlight for students the common structure of the functions.

Finally, an alternative to the add-on function is to insert an accessibility variable into the main CES function as follows:

$$U = [aL^\rho + bZ^\rho + c(1 - hD)^\rho]^{1/\rho} \qquad a, b, c, h > 0; \qquad 0 \neq \rho < 1. \tag{4C}$$

Here, distance affects utility differently compared with equation (4B), and the first order condition for optimum D is different. However, for any given D, the solutions L^* and Z^* are the same as in equations (9) and (10), which means that each step in an iteration over D is the same as in the previous paragraph. Again, values of c and h must be chosen carefully to avoid unrealistic results for the implied loss of utility from commuting; both c and h might be .05, for example. As usual, inspection of simulation results will help students learn how to judge the reasonableness of parameters. One caution: In equation (4C), when $D = 0$ the value of U is higher than the value of U in equation (4B), so that one cannot readily compare results from equation (4C) with results from equation (4B), even though for each form separately an iteration over D will reveal the global maximum.

CLASSROOM EXERCISES AND HOMEWORK PROBLEMS

Many simulations show special features of the model and also how the household's optimal choices respond to changes in parameters and exogenous variables. Obvious examples are responses to a change in income, a change in the size of α relative to β, and changes in other parameters of the utility or commuting cost function. The following are some other suggestions:

Transformation of Utility Function. Students may be confused to learn first that utility cannot be measured cardinally but then see a precise algebraic utility function. The teacher will have mentioned that a monotonic transformation of the function will not change the optimum. A simulation can demonstrate the principle convincingly by simulating transformations, including those that are complicated and that students do not recognize readily as monotonic.

Local vs. Global Maxima. The nonlinear budget constraint may lead to local maxima that are not the global maximum (in the Alonso model, local maxima sometimes occur at $D = 0$, for example). Even if one uses an optimization program that finds the global maximum reliably, it is good to show students just what a local maximum looks like, and here again, an iterative solution method is useful because it displays U for many nonoptimal distances.

Constrained Optimization. A pair of exercises can shed light on suboptimization. In the first exercise, the instructor (or student, in homework) changes a parameter—or perhaps income—that causes the household to change L and Z, but at the same time, he or she holds D constant, preventing the household from moving. In the second exercise, he or she allows the household to change D as well as L and Z. Splitting the problem into two parts helps clarify for students how a movement in space can improve welfare. The first exercise is a suboptimization exercise—the household optimizes the change in L and Z, subject to a given D. A clue that its result is not a true optimum is that $MMUD$ does not equal MCD.

THE BID-RENT APPROACH AND THE INDIRECT UTILITY FUNCTION

An instructor may want to introduce the *indirect utility function*, which has many applications that will show up in advanced courses. He can do so with an application of the Alonso model, namely the bid-rent approach, a feature of urban economics. It is another method to find the household's optimum location, alternative to the method already described. It is suitable only for experienced or highly motivated students, but it is potentially appealing if students are likely to take urban economics later, as in a city planning or real estate program. It was a prominent feature of Alonso's larger model of land prices (Alonso 1964, ch. 4), and it is now a staple of modern urban economic theory (Fujita 1989, 14–30).

Assume the household is at some location, specified by a value of D, and attains a level of utility, U_o. Then bid rent, P_L^{bid}, is the hypothetical dollar amount that is the maximum land price the household could pay without having U fall below U_o, assuming it optimizes its consumption of L and Z. The bid-rent function shows bid rent as a function of all possible values of D. There is a different bid-rent function for each level of U_o that one specifies in advance, so each bid-rent function is an equal-utility curve between rent and location; each curve is for a different level of U. The optimum location is the value of D at which a bid-rent function is tangent to the actual P_L function. This optimum location will be the same as the one determined by the standard theory described above, so the bid-rent approach is merely an alternative way to find the optimum location.

For a numerical example, return to the example of the modified Cobb-Douglas case where the household achieves $U = 2{,}572.197$. Holding U constant at

TABLE 1. Sample of Points on the Bid-Rent Function

D	Actual P_L	Assumed level of $V = U_o$	Solution for P_L^{bid}
0	2,400	2,572.197	1,410.47
1	1,455.67	2,572.197	1,157.23
2	882.91	2,572.197	833.58
3	535.51	2,572.197	535.42
3.06	519.69	2,572.197	519.69
4	324.80	2,572.197	311.73
5	197.00	2,572.197	166.80
6	119.49	2,572.197	83.05

U_o = 2,572.197, one can derive, as a function of D, the maximum land price the household could pay without falling below U_o, and that function will be the bid-rent function. The way to do this is to define the indirect utility function, V, which is the same as the U function except that equations (7) and (8) are substituted for L and Z, so that V now expresses the maximum utility the household can achieve, as a function of prices, income, and D. To calculate bid rent, one makes V a known number—in this numerical example, it equals U_o = 2,572.197—then solves for P_L, the only unknown in the V function. This solution for P_L is the bid rent; it is a hypothetical number and does not necessarily represent the market P_L. In the instructor's exposition, it is perhaps best to substitute P_L^{bid} for P_L in the V function to signal the distinction between the hypothetical number and the actual P_L. Solving the V equation for P_L^{bid} is mathematically equivalent to maximizing the land price the household could pay without its U falling below U_o.

I show a sample of points on the bid-rent function in this numerical example in Table 1; note that P_L^{bid} is always less than the actual P_L, except at the optimum location, where it equals P_L. Thus the optimum D is where the bid rent function is tangent to the actual P_L function.[4]

A TIME-ALLOCATION MODEL

It is possible to combine a time-allocation model, in the style of Becker (1965) and Michael and Becker (1973), with the Alonso model, enriching both models. Time allocation is not always a topic in the intermediate microeconomics course, but adding it to the Alonso model is a complication that may be useful for advanced students because it incorporates the opportunity cost for some households if commuting cuts into working time and reduces income. In the expanded model earned income is endogenous, depending in part on D. In contrast, in the model described previously, commuting reduces time in recreation, culture, and other activities but not working time, and income is exogenous.

Now there are four decision variables: land space, L; a composite good, Z; location, D; and hours of leisure, V. The choice of V and of D implicitly determine hours of work, which in turn determines money income. Leisure is the hours not working or commuting, so it includes sleep, basic personal and family maintenance, recreation, entertainment, and so forth. This portmanteau notion of leisure is a feature of the most basic time-allocation models in the literature.

The utility function is now

$$U = U(L, Z, V, D). \tag{11}$$

Commuting requires R hours each month, where R is a function of D (this function must be specified but is not shown here). The household spends C on commuting, and C is also a function of D (the function also must be specified). The household earns an hourly wage rate, w, which is exogenous and has Y' of unearned income, also exogenous.

Now the fundamental constraint is a time constraint, but time can be converted into money. If the household worked every hour of the month (assuming 30 days) and did not commute, it could spend this amount each month: $(1 - t)(720w + Y')$, where t is a personal income tax rate. This is called *potential income* in the literature. Allowing for commuting, the potential income is actually

$$J = (1 - t)[w(720 - R) + Y'] - C = (1 - t)(720w - wR + Y') - C. \tag{12}$$

The household allocates or spends this potential income on leisure, land space, and the composite good. In equation (12), potential income J equals the sum of potential hours of work, valued at $w(1 - t)$, plus after-tax unearned income, minus monetary cost of commuting. The price of an hour of leisure is an opportunity cost equal to $w(1 - t)$, so the amount of potential income allocated to leisure $= [w(1 - t)]V$. Therefore, the budget constraint is

$$w(1 - t)V + P_L L + Z = J. \tag{13}$$

Combining equations (12) and (13), one can write,

$$C + LP_L + Z = (1 - t)w(720 - R - V) + (1 - t)Y'. \tag{13A}$$

The left-hand side is money expenditure; the right-hand side is money income, so equation (13A) is an alternative way to write the budget constraint. Money income is endogenous because it depends on V, a decision variable, and on R, which itself depends on D, another decision variable.

The term $w(1 - t)$ appears often because it is the price (opportunity cost) of leisure, so define $w_t = w(1 - t)$. As will be seen, it is an important price in the extended model.

The marginal cost of distance, seen earlier as a key endogenous variable in the Alonso model, now includes an additional effect because distance increases R. Now it is

$$MCD = \partial(C + w_t R + P_L L + Z)/\partial D = dC/dD + w_t(dR/dD) + (dP_L/dD)L. \tag{14}$$

If iteration is used to find the optimum, the procedure is similar to the one before: At each possible D, the method uses equations to find optimal levels of three decision variables, L^*, Z^*, and V^*, then scans over values of D to find the value D^* that maximizes U. The equations for L^*, Z^*, and V^* at each D depend on the utility function that is assumed and are as follows (these are derived from first-order conditions for a maximum):

Cobb-Douglas Utility Function: As before, one can choose an additive function for the disutility of distance

$$U = AL^{\alpha}Z^{\beta}V^{\theta} - gD^{\gamma} \qquad A, \alpha, \beta, \theta, g > 0; \gamma > 1, \tag{11A}$$

or an accessibility multiplier form

$$U = AL^{\alpha}Z^{\beta}V^{\theta}[c(1 - hD)^{\gamma}] \qquad c, h > 0, 0 < \gamma < 1. \tag{11B}$$

In both cases, the solutions for L^*, Z^*, and V^* at each value of D are

$$L^* = [\alpha/(\alpha+\beta + \theta)](J/P_L), \tag{15}$$

$$Z^* = [\beta/(\alpha + \beta + \theta)](J/P_Z), \tag{16}$$

$$V^* = [\theta/(\alpha + \beta + \theta)](J/w_t). \tag{17}$$

CES Utility Function: Again, we can choose an additive form.

$$U = (aL^{\rho} + bZ^{\rho} + sV^{\rho})^{1/\rho} - gD^{\gamma}; \quad a, b, s, g > 0; \quad 0 \neq \rho < 1; \gamma > 1 \tag{11C}$$

or enter accessibility directly into the CES function as a good.

$$U = [aL^{\rho} + bZ^{\rho} + sV^{\rho} + c(1 - hD)^{\rho}]^{1/\rho} \quad c, h > 0. \tag{11D}$$

In both cases, the equations to solve for L^*, Z^*, and V^* at any given level of D are

$$L^* = [a^{\sigma}P_L^{r-1}/(a^{\sigma}P_L^r + b^{\sigma}P_z^r + s^{\sigma}w_t^r)]\mathrm{J}, \tag{18}$$

$$Z^* = [a^{\sigma}P_z^{r-1}/(a^{\sigma}P_L^r + b^{\sigma}P_z^r + s^{\sigma}w_t^r)]\mathrm{J}, \tag{19}$$

$$V^* = [a^{\sigma}w_t^{r-1}/(a^{\sigma}P_L^r + b^{\sigma}P_z^r + s^{\sigma}w_t^r)\mathrm{J}, \tag{20}$$

where $r = \rho/(\rho - 1)$ as before. In both the Cobb-Douglas and the CES cases, the algebraic similarity of the demand functions is revealing and can stimulate student discussion.

CONCLUSION

The Alonso model of a household's location choice is an example of the value of using computer simulation models in teaching microeconomics in a wide variety of educational situations. It enhances teaching basic concepts and also introduces realistic complexities without requiring mathematical background beyond the level of most intermediate students. Much of the value of this particular simulation model comes from the fact that it is based on an underlying theoretical model that is richer than the traditional models found in the textbooks accessible to most students; the underlying model has three or four, not just two, decision variables, it has a bad as well as goods, and it has an endogenous nonlinear price for one good. In addition to being a richer model, a computer simulation version of it frees up time for many more numerical examples in class and in out-of-class problems. If students study the numerical solutions carefully, it will help them gain some experience in evaluating a model for reasonableness of the functional forms and the parameters assumed, and this is one of the most important advantages.

The teacher can choose from possibilities of varying complexity. I have described four different utility functions and also two applications—the bid-rent approach and the time-allocation model—that are optional if one has especially able students and enough time available.

Of course, there are pitfalls to avoid. The teacher must not let students lose sight of the economics forest for the numerical trees, and some functions like Cobb-Douglas, CES, exponential, and power functions may cause sensitivity to small changes in some parameters. Corner solutions are possible. It takes care to get, in solutions, values of C, P_L, and other variables that are reasonable in the local context that students are familiar with. But the problems are ones that stimulate useful discussion and are well worth tackling.

NOTES

1. Alonso (1964) noted that one can either make Z the *numeraire* with price $= 1$ or define its price as an index whose value is not necessarily one (20). Most economists choose the former option.
2. In urban economics, the Alonso model is usually embedded in a more general monocentric city model, with all households commuting to the same central location. Some writers assume an exogenous value, P', for nonurban land that effectively places a lower floor to P_L, the residential land price. Whereas P' is exogenous to the household, it is the result of the general equilibrium process that determines land prices in all uses in all locations. In discussing the complexities of the process, Sinclair (1967) noted that nonurban land prices depend on much more than simply the transport cost of agricultural produce to a single market. However, in his leading treatise, Fujita (1989, 54, 94) made the simplification that agricultural land value is an exogenous constant and does not vary with distance from the city's center. Alonso, in his general equilibrium analysis (ch. 5, 86–87), also had an example in which agricultural rent is constant at every distance from the center. Fujita and Alonso made the assumption for expository convenience, but one might justify it if nonurban land is agricultural, forestry, or recreation land, and its value depends on demands in a region much larger than a single city, so that the value is not noticeably affected by distance within the city. Sinclair (1967) noted, for example, "There is rarely such a thing as a single local market, but rather a nationwide, or worldwide market" for some agricultural products (76). If one does assume a floor to P_L, then the distance at which the floor becomes operative is what Alonso (1964) called the "edge of the urban sector" (81) and Fujita (1989) called the "urban fringe distance" (56). It is a sharp edge because no urban household will locate beyond it, but, of course, such a rigid edge is unrealistic, and it seems an undesirable complication in a model used in a general microeconomics course.
3. The basic definition of the CES function is

   $$U = [aX^{(\sigma-1)/\sigma} + bY^{(\sigma-1)/\sigma}]^{\sigma/(\sigma-1)},$$

   where σ is the elasticity of substitution. However, almost every writer finds it convenient notationally to define the parameter $\rho = (\sigma - 1)/\sigma$, and then rewrite the function: $U = (aX^{\rho} + bY^{\rho})^{1/\rho}$ (Dixit 1976, 9–10; Varian 1992, 19–20, 112; Mas-Collel, Whinston, and Green 1995, 97; Jehle and Reny 1998, 130–32). In older textbooks, authors sometimes defined the parameter ρ as $(1 - \sigma)/\sigma$, in which case, the function becomes: $U = (aX^{-\rho} + bY^{-\rho})^{-1/\rho}$.
4. In an advanced course, still another option is to analyze the household's optimum by using the *expenditure function,* another useful concept (see Fujita 1989, 17–18).

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