Available online at www.sciencedirect.com
ScienceDirect
Physica A 384 (2007) 613–627
PHYSICA A
www.elsevier.com/locate/physa

A dynamic model for city size distribution beyond Zipf's law

Lucien Benguigui[a,*], Efrat Blumenfeld-Lieberthal[b]

[a]Solid State Institute and Department of Physics, Technion-Israel Institute of Technology, 32000 Haifa, Israel
[b]Faculty of Architecture and Town Planning, Technion-Israel Institute of Technology, 32000 Haifa, Israel

Received 25 January 2007; received in revised form 15 May 2007
Available online 29 May 2007

Abstract

We present a growth model for a system of cities. This model recovers not only Zipf's law but also other kinds of city size distributions (CSDs). A new positive exponent α, which yields Zipf's law only when equal to 1, was introduced. We define three classes of CSD depending on the value of α: larger than, smaller than, or equal to 1. The model is based on a random growth of the city population together with the variation of the number of cities in the system. The striking result is the peculiar behavior of the model: it is only statistical deterministic. Moreover, we found that the exponent α may be larger, smaller or equal to 1, just like in real systems of cities, depending on the rate of creation of new cities and the time elapsed during the growth. It is to our knowledge the first time that the influence of the time on the type of the distribution is investigated. The results of the model are in very good agreement with real CSD. The classification and model can be also applied to other entities like countries, incomes, firms, etc

PACS: 89.65.−a; 05.20.−y; 05.40.Jc

Keywords: City size distribution (CSD); Power law; Exponent α

1. Introduction

Since the work of Auerbarch [1] in 1913 and Zipf in the 1940s [2] the city size distribution (CSD) has attracted the attention of scholars in many disciplines such as geography, economy, demography, and physics. At present, there is extensive literature on this subject. The linear relation between the size (e.g. population) of cities and their ranks on a log–log plot is often found to be a power law. The absolute value of the slope of this linear equation is the exponent $\mu (S \propto R^{-\mu})$. This distribution is also known as a Pareto distribution, since the function giving the number of cities with size equal to or larger than a given S is $P(S) \propto S^{-1/\mu}$. In the past, it was accepted that the only observed value of μ (for the CSD) is 1, but previous work showed that the value of μ ranges from 0.6 to 1.5 [3]. The power law and Zipf's law in particular have been largely accepted as a paradigm for the CSD.

*Corresponding author.
E-mail address: ssgilles@techunix.technion.ac.il (L. Benguigui).

0378-4371/$ - see front matter © 2007 Elsevier B.V. All rights reserved.
doi:10.1016/j.physa.2007.05.059

Nevertheless, there are several countries in which a different expression (non-linear) fits the log–log plot of the CSD better than a linear equation [4]. The number of these countries is too large to treat them as "exceptions" of the power law. Previous work has proposed the power law as a first approximation for the CSD; however, this approximation cannot be applied to the non-linear distributions. Based on the above, we proposed a new classification for the CSD that include both linear and non-linear distributions [13].

Several theoretical models have been elaborated to explain the CSDs. The "simple" models used only a limited number of parameters while economic models are based on a relatively large number. The theoretical "simple" models focus on a steady state distribution that corresponds to a power law with an exponent $\mu = 1$ (at the exception of Reed's model [6]). These models, that are either analytical or computer simulations, can be divided into two classes. The models in the first class are based on constant number of cities with equal sizes of population in the initial state, and then propose different mechanisms to redistribute the population between the cities [5]. The models in the second class propose mechanisms for the city growth, based on a varying number of cities (N) during the growth process [6–8]. Despite the fact that these models use different mechanisms to describe the growth of N, the power law is recovered.

The economic model of Gabaix [9] can be classified into both of these classes. It is based on a city growth following the Gibrat law [10] and on a minimum value of the smallest city which is equal to a fraction of the mean size of the ensemble of cities. The latter mechanism influences the redistribution of the sizes of the cities. The economic model of Rossi–Hansberg and Wright [17] are worth mentioning as well, as it yields CSDs which do not necessarily fit the Pareto distribution.

The aim of this paper is to propose a model which can describe all (or most of) observed cases of CSDs; the linear (with $\mu = 1$ or $\neq 1$) case as well as the non-linear cases. Preliminary results were already presented in Ref. [11]. Before exposing our model and its characteristics, we present, in Section 2, our classification of CSD based on a new exponent [11,13]. The model is explained in Section 3 followed by the results in Section 4. We compared the present model with some other models showing its advantages (Section 5). Finally, in Section 6, we compare the results of the model with the CSD of the USA.

2. A new classification based on a new exponent

After considering a large number of the countries and other entities which can be quantified by a "size" (e.g. incomes, animal species, etc.) we found [11] a general mathematical expression which allows to describe all known cases of CSDs: ($y = \mathrm{Ln}\, S$ and $x = \mathrm{Ln}\, R$):

$$y = y_0 - H(\alpha)\mu[b + H(\alpha)x]^{\alpha}, \tag{1}$$

where S represents size, R represents rank, α is the new positive exponent, y_0, μ, and b are parameters, and $H(\alpha)$ is a step function equal to -1 if $\alpha < 1$ and to 1 if $\alpha \geqslant 1$. Eq. (1) can be written as $y = y_0 - \mu[b + x]^{\alpha}$ when $\alpha \geqslant 1$ and as $y = y_0 + \mu[b - x]^{\alpha}$ when $\alpha < 1$. In both cases, Eq. (1) describes a concave function which is a "parabola like" curve. When $\alpha < 1$, the symmetry axis of the parabola is parallel to the x-axis and when $\alpha > 1$ the symmetry axis of the parabola is parallel to the y-axis. A power law is recovered when $\alpha = 1$. In this case, μ is the Zipf exponent.

Thus, one can classify the CSD (and clearly other size distributions of different entities, [4b, 11]) and the mathematical expressions proposed by several authors for describing them, into three classes:

Class 1: Corresponds to a power law when the exponent α is equal to 1. The rank size distribution (or more precisely its plot on a log–log scales) of this class is a linear equation, meaning the relation between S and R is a power law.

Class 2: Is defined by $\alpha > 1$. The two expressions proposed by Laherrere and Sornette [4b]: a fractal parabola and the stretched exponential equation belong to this class.

Class 3: Is characterized by $0 < \alpha < 1$ and the lognormal distribution with $\alpha = 0.5$ belongs also to this class.

The distribution functions of classes 1 and 2 present a quasi-divergence for the small sizes. This, however, is not the case for class 3. If α is smaller than 1, the distribution function exhibits a maximum for the small sizes (as for a lognormal distribution) or a finite value. An example for this can be found in the size distribution of the retail system of Israel for which the distribution function is exponential [12].

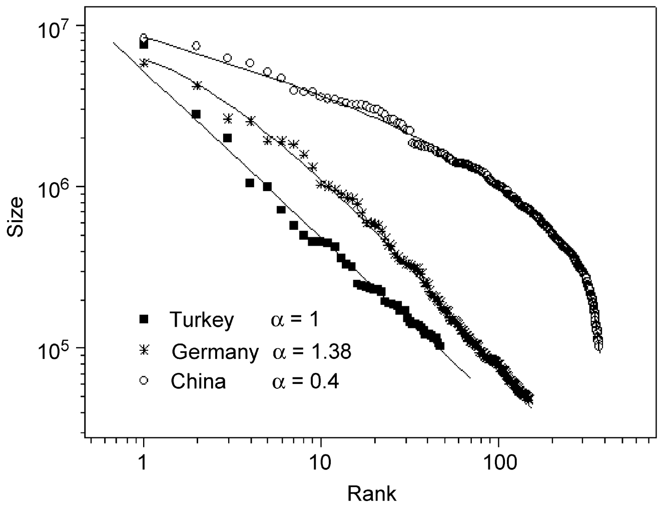

Fig. 1. Log–log rank size plots for China, Germany, and Turkey.

We have recently analyzed 41 cases of CSDs [13] and found that for 17 cases $\alpha = 1$ and μ varies from 0.61 to 1.25; for 13 cases $\alpha > 1$ and varying from 1.3 to 2.59, and for one case $\alpha = 0.41 < 1$ (Fig. 1). The other cases correspond to non-homogeneous distributions, i.e. these distributions could not be described by a unique formula.

3. The model

The model we propose here was configured in order to recover all the preceding cases of CSDs without looking only for a steady state. It uses similarly to Simon [7], Gabaix [9], Reed [6], and Blank and Solomon [8], a random city growth. The economic justification for this model is based on the Gibrat law [10]. This means that the cities of a specific country grow randomly with the same rate distribution. It is commonly accepted that the distribution type is not very important.

The presented model is a generalization of the model of Blank and Solomon [8]. It is a computer simulation which has some advantages on an analytic one. It is worth mentioning that strictly speaking, a real distribution function for a given CSD does not exist. The actual functions are the cumulative distributions that present the number of the cities equal to, larger (or smaller) than a given size. For the small cities, the relative difference in the population between two successive cities is very small, thus, these cumulative distributions can be considered continuous and derivable. For the large cities, on the other hand, the cumulative functions are stepwise and not derivable. Thus, the distribution function which is the derivative of the cumulative function (in absolute values) is not defined for the large cities. Due to this reason, a computer model which is not based on continuity is especially well adapted to the CSD.

Our model belongs to the "simple" models and it is based on the following four stages:

1. We begin with N_1 cities (100 or 200), with the same initial population $S_1 = 1$.
2. At each step T, a city is picked at random. The size $S_i(T)$ of the city i is changed at the step $T+1$ to:
$$S_i(T+1) = \gamma S_i(T), \tag{2}$$
where γ is a random variable with a uniform distribution between $0 < \gamma_m < 1$ and $\gamma_M > 1$. The values of γ_m and γ_M are chosen so that the mean value, given by $\gamma' = (\gamma_m + \gamma_M)/2$, is slightly larger than 1. In the presented simulations we took $\gamma' = 1.015$.
3. If the size of a city decreases below the initial value 1, the city disappears.
4. After K steps a new city is introduced with an initial size $S_1 = 1$. The former rule applies to the new cities as well, and if their size decreases below 1, they disappear.

The growth described by (2) is a multiplicative random walk and was extensively studied by Solomon and coworkers [8,16]. It was shown in particular that for long periods of time this growth process tends toward a lognormal distribution if the number of cities goes to infinity.

Due to the condition in point 3, after several steps the number of cities drops rapidly below the initial N_1 (before that a new city appears). The result of the above condition implies that the initial number of cities is a random variable. The probability p_s for a city to survive was determined by means of simulations. We got a mean value of p_s which is approximately 0.25. Due to stage 3, the mean rate of the growth of N is not K^{-1} but $p_s K^{-1}$ and in fact, the number of the cities N during the growth is also a random variable. In Appendix A, we present the evaluation of p_s for a growth model, only slightly different from ours, but easier to solve. We found in this latter case that $p_s = 0.32$, which is very close to the value we found for our original p_s.

The model is novel in two aspects. The first is connected with stage 4 that was mentioned above. Other models use the same growth process associated with a variation in the cities' number. They adopt, however, a definite rate which does not change during the growth. In the present model, based on specific situations, K can be either a constant or a function of time (i.e. the number of steps in the program, see further explanation below). Furthermore, as opposed to previous models (with the exception of [6]) the introduction of new cities is a random process. This makes the model very flexible and it can be adapted to different situations characterized by a different function $K(T)$. The second aspect of novelty is that unlike previous models, we do not look for a steady state but consider the time dependent state.

We cannot use the number of steps T to define the time in this model, since the growth of one city as a function of T will depend on the total number of cities. After a number of steps ΔT the average number of times that each city is picked at random and experiences a change in its population is $\Delta T/N$. We adopt for the time interval Δt which corresponds to the step interval ΔT the relationship $\Delta t = \Delta T/N$.

Thus, if there is a change in the number of cities by one unit after K (supposed to be constant) steps, the time t after T steps is given by

$$t = K/N_1 + K/(N_1+1) + K/(N_1+2) + \cdots + K/(N_1 + T/K) \tag{3a}$$

or

$$t = K\sum_j \frac{1}{(N_1+j)} \quad \text{for } j \text{ between 0 and } \frac{T}{K}. \tag{3b}$$

N_1 is the initial number of cities. In Appendix B, we show that:

$$t = K\,\mathrm{Ln}\left[1 + \frac{T}{(KN_1)}\right]. \tag{4}$$

We used also the simulations to verify that (3) is correct. From (4), one can get T as a function of t, thus:

$$T = KN_1[\exp(t/K) - 1]. \tag{5}$$

Since the number of cities is proportional to $T(N = p_s(N_1 + T/K))$, when K is taken as a constant, the variation of N is exponential with time [18]. But when K is taken as a function of T, other relations between T and t can be constructed [19].

We ran the simulation for a given choice of K and stop after a number of steps T. The model determined the CSD from the populations of the cities and plotted the log–log rank size distribution. Thereafter, we used the software package of Origin® to find the values of the parameters in Eq. (1) and thus the mathematical approximation of the distribution.

4. Results

As mentioned above, the flexibility of the model is in the possibility to change the rate of introducing new cities. In this present work we examined two particular situations: K is constant during the growth, or K decreases following a step function. Qualitatively speaking, both options (constant or varying K) yield very similar results. Nevertheless, the results of the latter case, as shown below, present good correlation with the real cases of CSD. Thus, we present the results of the latter case only. We took K as constant until $T = 25\,000$

and then it was divided by 2 for $T > 25\,000$. The calculation of the time t is made with the help of (3):

$$\text{if } T < 25\,000, \quad t = K \,\mathrm{Ln}\left(1 + \frac{T}{KN_1}\right), \tag{6a}$$

$$\text{if } T > 25\,000, \quad t = K \,\mathrm{Ln}\left[1 + \frac{(25\,000)}{KN_1}\right] + K/2 \,\mathrm{Ln}\left[1 + \frac{2(T - 25\,000)}{KN_1}\right]. \tag{6b}$$

Again, T is an exponential function of t and the number of cities also presents an exponential dependence on the time with two time constants: K at the beginning and $K/2$ at the last part of the growth.

The model is only statistical deterministic. In other words, if one chooses the value of the number of steps T and the function $K(T)$, one has only a certain probability to get the output values which are the parameters of the Eq. (1). Thus, one needs to perform several realizations in order to find the distribution of the parameters for given values of starting parameters (T and $K(T)$).

4.1. *The exponents α and μ as a functions of t and K*

The main results of the model show that the CSD is dependent of two parameters: the time of the growth and the value of K in the first part of the growth, for a given distribution of the rate growth γ. We examined different values of K from 25 to 1000 and different values of time from 50 to 400. The best way to visualize the results is to plot a "phase diagram" $t-K$ in which the space is divided into three regions that correspond to the three classes of the CSD (Fig. 2). The boundary between the region corresponding to $\alpha > 1$ and the one corresponding to $\alpha < 1$ is not a defined line due to the non-deterministic character of the model. This boundary is in fact a very narrow region where α may have values that belong to different classes (larger than 1 or smaller than 1, but not equal to 1).

As we would like to focus on the region corresponding to $\alpha = 1$, we will start this discussion with region 3 that corresponds to $\alpha < 1$. The values of α in this region resulted from choosing large K (i.e. small rate of new cities) or/and relatively small times. In this region the system remains analogous to one without creation of new cities. Fig. 3 presents the histogram of the values of α for $K = 1000$ and $t = 200$ showing the distribution being very close to a Gaussian one. In the inset of Fig. 3, the log–log rank size plot is presented, with the characteristic shape of this class: a very large slope for the small sizes. Note that the shape does not change too much with t.

Region 2 (where the values of $\alpha > 1$) corresponds to intermediate values of K and also not too large values of time. Fig. 4 presents the log–log rank size plots at different times for $K = 200$. In Fig. 5, α is plotted as a

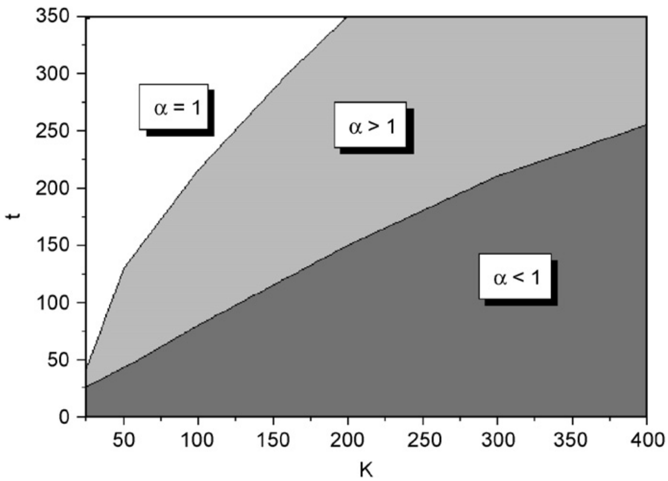

Fig. 2. Phase diagram time–rate of city growth. One distinguishes between three regions corresponding to the three classes: class 1 or $\alpha = 1$, class 2 or $\alpha > 1$ and class 3 or $\alpha < 1$.

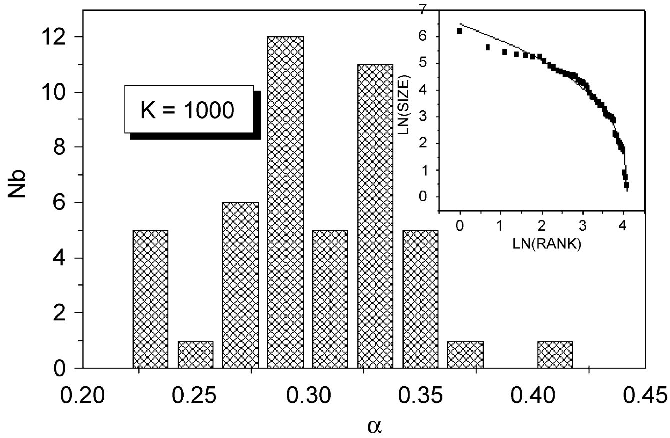

Fig. 3. Histogram of α for different realizations, $K = 1000$ and $t = 250$. In the inset a log–log rank size plot.

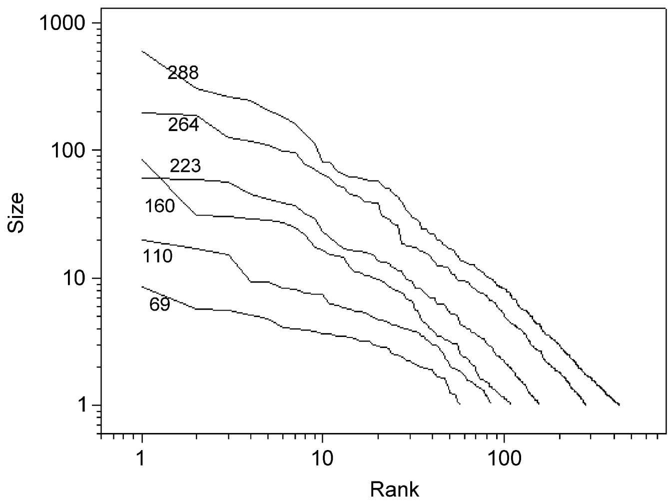

Fig. 4. Log–log rank size plots for $K = 200$ at different times.

function of t and the transformation from $\alpha < 1$ (region 3) to $\alpha > 1$ (region 2) can be seen. The histogram of the values of α is shown in Fig. 6 for $K = 200$ and $t = 230$, i.e. in region 2.

The definition of the region 1 ($\alpha = 1$) is slightly different from the others. In this region, for different realizations corresponding to one point (or to a couple of values of t and K) there is at least one which yields $\alpha = 1$. The case $K = 200$ and $t = 380$ is an example for this behavior. Fig. 6 presents the plot of two α histograms, one for $K = 200$ at $t = 230$, and the other at $t = 380$. For the smaller value of time, all the values of α are larger than 1 (class 2) but for the larger value of time some realizations yield $\alpha = 1$ (class 1) and others $\alpha > 1$ (class 2). This means that in this region one can obtain the system in either class 1 or 2. The number of realizations, with $\alpha = 1$, however, increases if t increases and/or if K decreases. Roughly speaking, this region corresponds to small K or/and large t. Fig. 5 presents the variation of α with t for $K = 100$ and 50 and one sees that α reaches the value 1 after beginning with a value different than 1, smaller than 1 for $K = 100$ and larger than 1 for $K = 50$.

In the realizations where $\alpha = 1$, the exponent μ is dependent on the realization itself, but for a given realization it remains practically constant with the time, once the system reached $\alpha = 1$. In fact, considering the different realizations for K between 50 and 200, μ varies from 0.59 to 1.49, in excellent correlation with the real data for countries with $\alpha = 1$. Fig. 7 presents the mean value of μ as a function of K. In Fig. 8 we show the histograms of the exponent μ for three cases: (a) $K = 100$ and $t = 229$ when only a small fraction of the

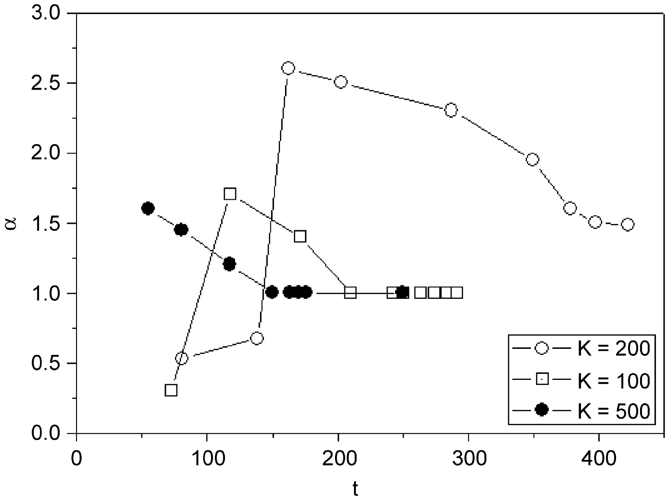

Fig. 5. Variation of the exponent α with the time for three different values of K. Note that for $K = 50$ and 100 the exponent reaches the value 1.

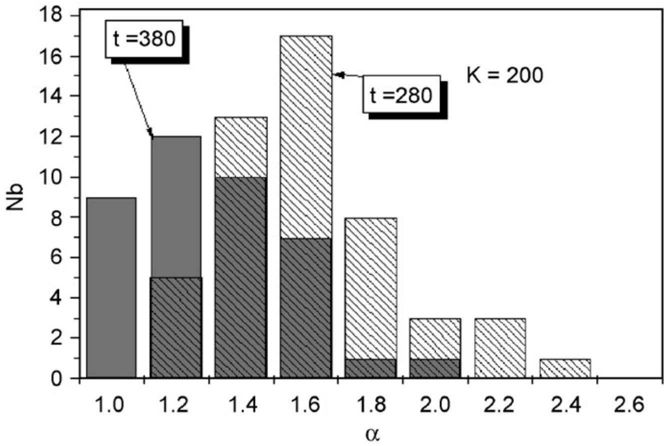

Fig. 6. Histograms of α for $K = 200$ at two different times: $t = 280$, all the realizations give $\alpha > 1$; $t = 380$ some realizations give $\alpha > 1$ and others $\alpha = 1$.

realizations yields $\alpha = 1$; (b) $K = 100$ and $t = 288$ when the majority of the realizations yield $\alpha = 1$ and (c) $K = 200$ and $t = 380$ when only some realizations give $\alpha = 1$.

To understand the relation between μ (mean) and K (Fig. 7), we consider the relation $S_{\min} = 1 = S_{\max}/(N_t)^{\mu}$ from which one deduces that

$$\mu = \mathrm{Ln}(S_{\max})/\mathrm{Ln}(N_t). \tag{7}$$

$S_{\max}$ can be expressed as $S_{\max} = S_0 + S_1 \exp(\Gamma t)$ and $N_t = N_0 + N_2 \exp[(2/K)t]$ where $\Gamma = \gamma_M. - 1$. After performing some simple calculations this equation can be written as

$$\mu = [K(\Gamma + A)]/(2 + BK), \tag{8}$$

where A and B are dependent on the parameters appearing in the expressions of $S_{\max}$ and N_t. This also means that μ is slightly dependent on t, although it is difficult to detect this dependence. We fitted the curve $\mu(K)$ of Fig. 7 by expression (8) and got a good agreement between the data and the theoretical expression [15].

The results suggest that, at least for not too large values of K, the system will always reach region 1 meaning $\alpha = 1$ independently of the realizations. One can call this state a quasi-steady case because α and μ are

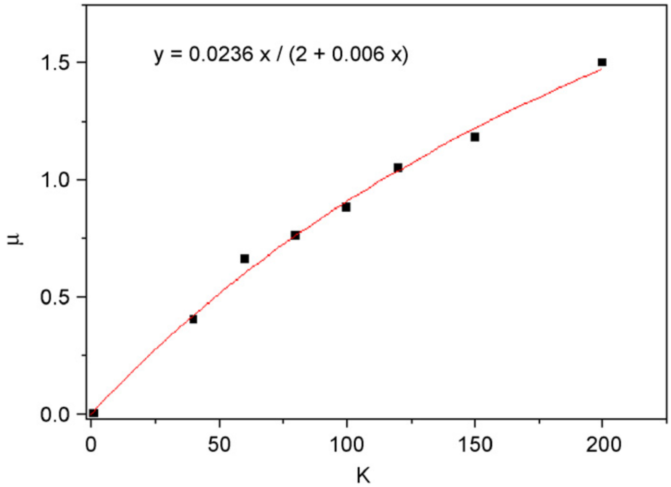

Fig. 7. Variation of the mean value of the exponent μ with K. Note the good fit with expression (8).

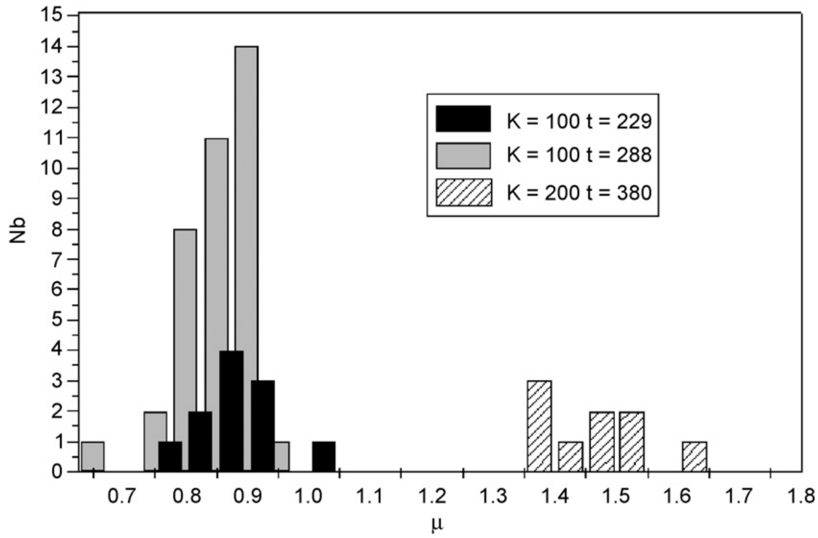

Fig. 8. Histograms of μ for three cases: $K = 100$, $t = 229$ (only some realizations give $\alpha = 1$); $K = 100$, $t = 288$ (almost all the realizations give $\alpha = 1$) and $K = 200$, $t = 380$, for the realizations giving $\alpha = 1$.

constant. We believe, however, that the most important aspect of this model is that it yields all three classes defined above (based on the analysis of real distributions) and that the values of α and μ are strongly correlate to real CSD. Finally, the time factor was not taken into account in other models. However, we would like to emphasize its importance in the formation of a given CSD.

4.2. The case $K = 100$, $\alpha = 1$

In this section we elaborate on the case of $\alpha = 1$ in order to make a qualitative comparison with a real situation (see further discussion in Section 6).

In Fig. 9 we show the log–log rank size plots of the same realization of the model for different times. The inset in this figure presents the variation of α and μ with t. When the system reaches the quasi-steady state ($\alpha = 1$), μ is practically constant.

The change of the number of cities with time is presented in Fig. 10. The function that describes the cities growth can be interpreted as compound of two parts. For small t ($T<25\,000$) the growth is linear (although rigorously it is an exponential growth as seen above) but further ($T>25\,000$) the curve can be fitted, as expected, by an exponential with time constant $K/2 = 50$.

A comparison between the log–log rank size plots of different realizations at the same time ($t = 229$ in Fig. 11) can show how the system evolves toward the quasi-steady state. One notes that the first distribution is a power law with an exponent 0.81, the second has a primate city and $\alpha>1$, and the third is a concave function ($\alpha>1$) with a parabola like properties. The model often yields a homogenous distribution except for the largest city which does not belong to this distribution. This largest city, often known as the primate city, can be either above or below the function that represent the distribution of the rest of the cities. The remarkable aspect, however, is that for $R>10$ (about 95% of the cities) the distribution hardly changes in different realizations. The influence of the different realizations on the distribution is mostly reflected in the largest cities.

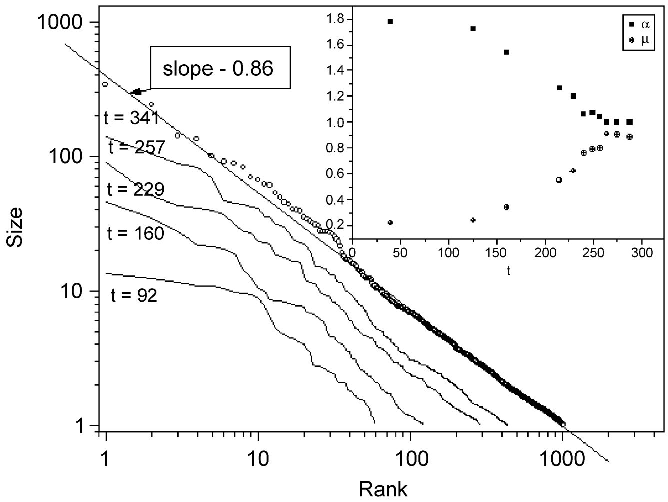

Fig. 9. Log–log rank size plots for $K = 100$ at different times. In the inset variations of α and μ.

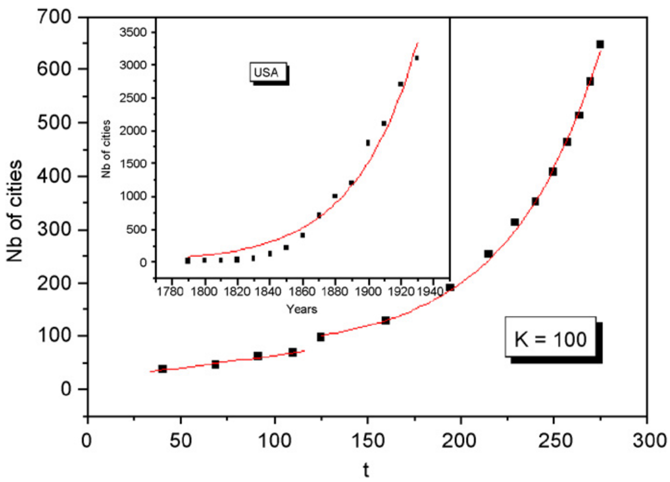

Fig. 10. Number of cities as a function of time. Note the two parts of the curve. In the inset, exponential variation of the number of USA cities with time. The variation of the number of cities is very similar in the model and in the case of the USA cities.

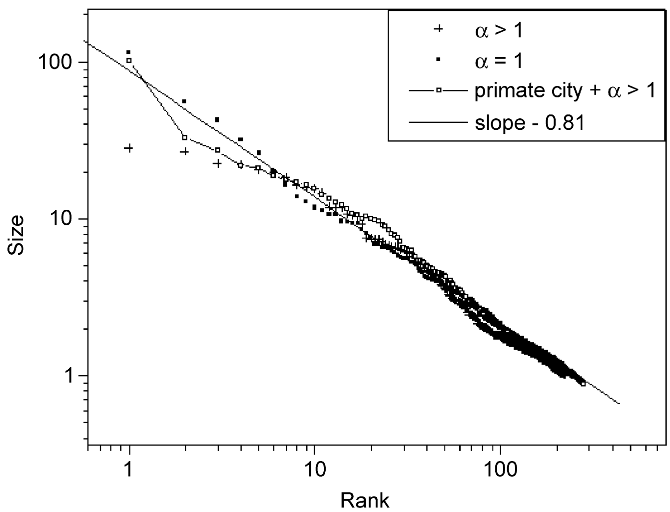

Fig. 11. Log–log rank size plots for $K = 100$ and $t = 229$ for three different realizations giving three different results.

5. Comparison with existing models

In this section we compare our model with some other models based on the two factors: (a) the growth of the cities and (b) the variation in the number of cities.

5.1. Simon's model

In this analytical model [7], new migrants arrive at each period of time and either they reach an existing city or form a new one. The probability to go to a particular city is proportional to its size and the probability to form a new city is equal to a parameter π. After some time, the distribution converges to a power law with an exponent $\mu = 1 - \pi$. Since the rate of creating new cities is small, the probability π is typically small and the exponent is practically equal to 1. Krugman [14] considers this model as a very successful one for its simplicity. This model, however, has several drawbacks. First, it implies in the frame of the model itself, that the ratio between the size S of a given city and the total population is constant. It is a strong hypothesis which has not been checked. Second, the main criticism is that the model can describe only a limited number of countries: those with $\alpha = 1$ and $\mu = 1$.

5.2. Reed's model

This model [6] which has been solved analytically is based on the following factors concerning the creation of new cities: (A) The time of the creation of a new city is a random variable and the mean number of new cities increases exponentially as $\exp(\lambda t)$ and (B) the initial population of a new city X_0 is a random variable with a lognormal distribution. Contrary to Reed's claim, his model yields the steady state since the time does not appear in the final results. The shape of the distribution depends on an exponent β which can be larger or smaller that 1. If $\beta > 1$, the distribution exhibits a maximum which corresponds in our classification to class 3 ($\alpha < 1$). If $\beta < 1$, the distribution diverges on the side of small cities and belongs to class 2. Reed claims that Zipf's law can always be observed on the tail of the large cities with an exponent which is not necessarily 1. This model seems very successful since it can describe all types of CSD in our classification.

A closer examination of this model's results, however, reveals that the model does not correlate to our results. When looking for relations between the parameters of the model in order to get $\beta > 1$ or < 1, one can find the following condition: if the mean rate of the city growth is γ (Reed uses the letter μ) and σ is a parameter introduced by Reed and is "reflecting the variability in the growth rate", then $\beta < 1$ if $\lambda < \gamma + \sigma^2$. This means that class 2 is obtained for a small rate of creation of new cities and conversely class 3 is obtained for a large rate. This fact is in variance with our model in which, for large values of K (or in the Reed's notation, small values of λ) the CSD corresponds to class 3, while for small values of K (or large values of λ) the CSD corresponds to class 2. Our results seem more reasonable since in the case of $K \to \infty$ (no new city is created) the distribution is near a lognormal one. Only the introduction of new cities to the system changes a distribution with a maximum point (e.g. like a lognormal one) to a power law, or in other words from $\alpha < 1$ to $\alpha > 1$ and $\alpha = 1$.

The difference between our model and Reed's stands in the choice of the initial population of a new city and in the fact that there is no minimum size for a city. Reed's choice implies that sometime, the initial population of a new city is very large or very small contrarily to the condition in our model. This might be actually the mechanism behind the particular results of Reed's model.

5.3. *Blank and Solomon's model*

As mentioned above Blank and Solomom's model [8] is a computer one. The difference between the models appears only in the way in which new cities are introduced. In their model the rate of creation of new cities $\mathrm{d}N/\mathrm{d}t$ is proportional to the derivative of the total population P. At each step of the program a number ΔN of new cities appears:

$$\Delta N = K'[P(T+1) - P(T)]. \tag{9}$$

This indicates a linear relationship between the number of cities and the total population. The CSD distribution, in the steady state, corresponds to Zipf's law, meaning the exponent $\mu = 1$, just like the authors expected. The authors have also mentioned that with extremely low value of K' one recovers the case that we refer as class 3. Despite the fact that the way of creation of new cities is different from our model, this model yields Zipf's law.

To understand this point, we examine the consistency of (9) with Zipf's law. If $S_i = S_{\max}/R_i^{\mu}$, the total population is

$$P = \Sigma S_i = S_{\max}\Sigma(1/R_i^{\mu}). \tag{10}$$

If $\mu = 1$, the sum is (for large enough values of R_i) equal exactly to $\gamma_E + \mathrm{Ln}(N_t)$ since the largest R represents the total number of cities (γ_E is the Euler constant approximately equal to $= 0.755215\ldots$). If $\mu \neq 1$, the sum can be approximated by an integral and one finally gets for P, taking into account that $S_{\max} = N_t^{\mu}$ (since we took for the smallest city $S_{\min} = 1$):

$$P = N_t[\gamma_{E+}\mathrm{Ln}(N_t)] \quad (\mu = 1), \tag{11a}$$

$$P = (N_t - N_t^{\mu})/(1-\mu) \quad (\mu \neq 1). \tag{11b}$$

Rigorously speaking, the relation between P and N_t is not linear. Yet, for N_t varying from 50 to 1000 (which is the usual range of the number of cities for which Zipf's law is observed) the two expressions (11a) and (11b) (for values of μ between 0.8 and 1.2) can be well approximated by a linear relationship. The conclusion is that introducing condition (9) to the model yields Zipf's law but only as an approximation.

5.4. *Tuncay's model*

The model of Tuncay [20] is a computer simulation. Similarly to our model and that of Blank and Solomon, it is based on a multiplicative random process with introduction of new cities during the growth. At each step of the growth, a city may split in two (the already existing city and a new one) with a probability H and this is the process of creation of new cities. The population of the new daughter city is that of the mother city (say S) multiplied by $k(0<k<1)$ and the population of the mother city becomes $(1-k)S$. The result of this model is a CSD which belongs to class 3 (see Fig. 2). The probability density function exhibits a maximum, which can be easily shown as a property of the class 3. Apparently, this result is independent of the value of k. However, the values of H in the simulation, which is equivalent to $1/K$ in our model, are very low (order of 0.002).

Consequently, the system is in the region of large values of K, i.e. class 3 as indicated in Fig. 2. Probably, with larger values of H, the distribution might be different.

Tuncay compares his results with languages only and not with cities. This comparison yields a good qualitative agreement using the chosen values of the parameters. We would like, however, to point out that the process of creating new cities by splitting the existing ones does not seem to be very realistic. This, in our opinion, is the major weakness of the model.

6. Comparison with real CSD and the USA case

6.1. The exponents

As shown in Fig. 5, the exponent α varies from 0.3 to values larger than 2.5 including the case $\alpha = 1$. These are exactly the values of this exponent as observed in several countries [13]. Exponent μ (in class 1) can obtain values between 0.6 and 1.45, depending on K. Here also the values are those observed in reality. Thus, we can say that our model is very successful in terms of the values of the exponents.

6.2. The USA case

In order to strengthen the validity of our model we conducted a more rigorous comparison of its results with the dynamics of a CSD of a unique country. We took the data for the USA in the second book of Zipf quoted in Ref. [2]. The qualitative comparison of the model's results with the CSD of the USA is presented in the following figures: (a) Fig. 12: the log–log rank size plots are strongly analogous to those of the model for $K = 100$; (b) Fig. 12: the dependence of α and μ on time can be compared with Fig. 5; and (c) Fig. 10: the change in the total number of cities with time, which can be approximated by an exponential equation.

To summarize, there is a good qualitative agreement between the results of the models and the dynamics of the CSD of the USA. We believe this fact strongly supports the validity of our model.

7. Conclusion

We have developed a model of city growth based on two basic factors: a random multiplicative growth and an increase in the number of cities. Contrarily to other models we can make two statements: first, the method of increasing the number of cities can be changed and adapted to different situations. Second, we investigated

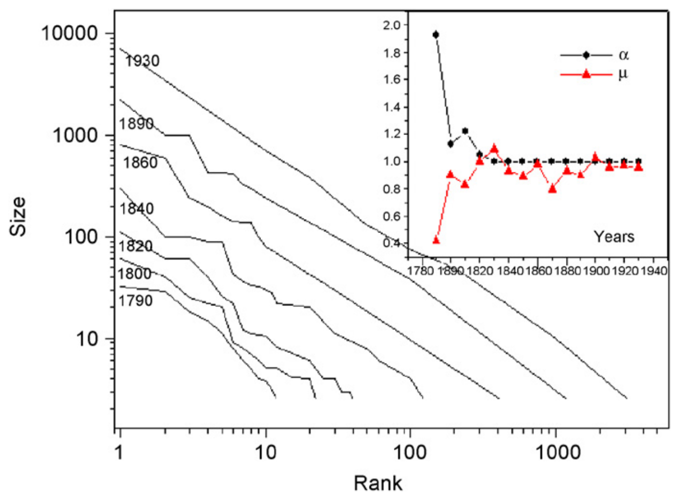

Fig. 12. Log–log rank size plots for the USA cities (data from Ref. [2b]). In the inset, variations of α and μ with time.

the influence of the time on the system and found it crucial. Based on a particular choice of increasing the number of cities, defined by the parameter K, we got the following results:

1. With the explicit introduction of the time, the model recovers all known types of CSDs, which we characterized by a positive exponent α smaller than, larger than, or equal to 1 (i.e. Zipf law). For large values of time and large growth rates of the number of cities, the system reaches a quasi-steady state. Although this point was postulated by previous models, our model is the first to effectively consider it.
2. The values of the new exponent α and the exponent μ are in excellent agreement with the real values, measured in different countries, of the growth rates of the city size and of the number of cities.
3. The model is only statistical deterministic; for a particular choice of time and K, the model does not yield definite values of α and μ but only certain probabilities. It is to our knowledge the first time that this property is reported in models of city growth and this fact is also valid for the quasi-steady state. One cannot exclude the possibility that for very large times and consequently for very large number of cities this indeterminacy will disappear. Real systems, however, have a finite number of cities thus the non-deterministic character of the model is completely relevant.
4. The mean values of the exponents can be determined and lead to the following results: $\alpha<1$ if the growth rate of the number of cities is small. When this rate increases the values of α changes to $\alpha>1$, and finally, for large values of time and/or large growth rate of the number of cities $\alpha=1$. The mean value of the exponent μ decreases when this growth rate increases.
5. It is to our knowledge the first time that the results of a model of city growth are compared with the dynamics of the CSD of one country. We have found a strong correlation between the results of the models and the size distribution of the USA cities between 1790 and 1930.
6. Finally, the fact that the model is not strictly deterministic implies that one should be very cautious about the attempt to relate the values of μ to economic processes. Following our model, the variations of μ and α are associated with subtle changes in the randomness of the model. This randomness mimics the economic and political events in a given country. It might be considered that only the ranges of the values of α and μ may have meaningful interpretations.

Appendix A

From Eq. (2) one has:

$$\mathrm{Ln}[S(T+1) = \mathrm{Ln}[S(T)] + \mathrm{Ln}\,\gamma. \quad \text{(A.1)}$$

The logarithm of the population of a city can be seen as the position of a walker along a line where it jumps, at each step, from a position given by $\mathrm{Ln}[S(T)]$ to a new position $\mathrm{Ln}(S(T+1)] = \mathrm{Ln}[S(T)] + \mathrm{Ln}\,\gamma$. There is a probability $p_+ = (\gamma_M - 1)/(\gamma_M - \gamma_m)$ for $\mathrm{Ln}\,\gamma$ to be positive (and the walker jumps toward the positive side of the line) and a probability $p_- = (1-\gamma_m)/(\gamma_M - \gamma_m)$ for $\mathrm{Ln}\,\gamma$ to be negative (and the walker jumps toward the negative side of the line). The walker begins to move from $\mathrm{Ln}\,S = 0$ and will disappear when for the first time it goes to the negative side. The determination of the probability p_D for the walker to disappear is very complicated because $\mathrm{Ln}\,\gamma$ is not constant but a random variable.

It is much easier to calculate this probability if one supposes that $\mathrm{Ln}\,\gamma$ is constant. In this way, one can get an order of magnitude for p_D. In this case, the walk is a biased random walk. The probability p_D is composed of two parts p_1 and p_2, probabilities of two independent events: p_1 corresponds to the event in which, at the first step, the walker goes toward the negative side and p_2 corresponds to the event in which the walker begins to jump toward the positive direction (probability p_+), comes back for the first time to the point $\mathrm{Ln}\,S = 0$ (probability p_R) and goes toward the negative side (probability p_-). One has

$$p_D = p_1 + p_2 = p_- + p_+ p_R p_-. \quad \text{(A.2)}$$

The probability p_R is equal to 1 because, in a random work, the walker returns with certainty to its original position [21]. Thus, (A.2) becomes:

$$p_D = p_-(1 + p_+). \quad \text{(A.3)}$$

Taking the values of $\gamma_M = 1.13$ and $\gamma_m = 0.9$, one gets that $p_D = 0.68$ and this gives for the survival probability $p_S = 0.32$ not too far from the value 0.25 in our simulation.

Appendix B

We begin with Eq. (3b):

$$t = K\sum_{j=0}^{T/K}\frac{1}{N_1 + j} = Kt', \tag{3b}$$

with t' given by ($A = T/K$)

$$t' = \sum_{j=0}^{A}\frac{1}{N_1 + j}. \tag{B.4}$$

Eq. (B.4) can be written as (with $x = N_1 + j$)

$$t' = \sum_{x=N_1}^{N_1+A}\frac{1}{x}. \tag{B.5}$$

The sum of the right-hand side can be viewed as the difference of two sums:

$$\sum_{x=N_1}^{N_1+A} = \sum_{1}^{N_1+A} - \sum_{1}^{N_1}. \tag{B.6}$$

If N is large enough, the sum $\sum 1/x$ from $x = 1$ to N is equal to $\gamma_E + \mathrm{Ln}\,N$ where γ_E is the Euler constant approximately equal to 0.755215…. Thus, one has:

$$\sum_{1}^{N_1+A}\frac{1}{x} = \gamma_E + \mathrm{Ln}(N_1 + A) \tag{B.7}$$

and

$$\sum_{1}^{N_1}\frac{1}{x} = \gamma_E + \mathrm{Ln}\,N_1. \tag{B.8}$$

Finally one gets:

$$t' = \mathrm{Ln}\left(\frac{N_1 + A}{N_1}\right) = \mathrm{Ln}\left(1 + \frac{A}{N_1}\right) \tag{B.9}$$

and

$$t = K\,\mathrm{Ln}\left(1 + \frac{T}{KN_1}\right). \tag{4}$$

References

[1] F. Auerbach, Petermanns Geogr. Mitt. 59 (1913) 74.
[2] [a] G.K. Zipf, Human Behavior and the Principle of Least Effort, Addison-Wesley, Inv. Cambridge, 1949;
[b] G.K. Zipf, National Unity and Disunity, The Principia Press, Bloomington, 1941.
[3] K.T. Rosen, M. Resnick, J. Urban Econ. 8 (1980) 165.
[4] [a] G. Alperovitch, J. Deutsch J. Reg. Sci. 35 (1995) 267;
[b] J. Laherrere, D. Sornette, Eur. Phys. J. B 2 (1998) 525.

[5] [a] F. Sembolini, F. Leyvraz, Physica A 352 (2005) 612;
[b] H. Zanetti, S.C. Manrubia, Phys. Rev. Lett. 79 (1997) 523;
[c] M. Marsili, Y.C. Zhang, Phys. Rev. Lett. 80 (1998) 2741.
[6] W.J. Reed, J. Reg. Sci. 42 (2002) 1.
[7] H. Simon, Biometrika 42 (1955) 425.
[8] A. Blank, S. Solomon, Physica A 287 (2000) 279.
[9] X. Gabaix, Q. J. Econ. 114 (1999) 739.
[10] R. Gibrat, Les inégalités économiques, Paris, 1931.
[11] L. Benguigui, E. Blumenfeld-Lieberthal, Int. J. Mod. Phys. C 17 (2006) 1429.
[12] M. Marinov, Ph.D. Dissertation, Technion, 2004.
[13] [a] L. Benguigui, E. Blumenfeld-Lieberthal, Proceedings of CUPUM05, 2005, ⟨http://128.40.59163/cupum/search/Papers/detail.asp?|d = 419⟩;
[b] L. Benguigui, E. Blumenfeld-Lieberthal, Comput. Environ. Urban (2007), doi:10.1016/j.compenvurbsys.2006.11.002.
[14] P. Krugman, The Self-Organizing Economy, Blackwell, Cambridge, 1996.
[15] To ameliorate the fit, we added (in Fig. 7) the point (1,0). If $K = 1$ there is creation of a city at each step and the number of cities grows rapidly, much faster than the growth of the city sizes. Therefore, the log–log rank size plot is a line parallel to the rank axis, i.e. $\mu \approx 0$.
[16] Z.F. Huang, S. Solomon, Physica A 294 (2001) 503.
[17] E. Rossi-Hansberg, M. Wright, 2005, ⟨www.nber.org/papers/w11202⟩.
[18] It is also possible to find the exponential growth of the number of cities with the help of the two relationships: $\mathrm{d}t = \mathrm{d}T/N$ and $\mathrm{d}N/\mathrm{d}T = 1/K$.
[19] In a forthcoming paper we use K as an explicit function $K(T)$ in order to apply the model to the systems of cities of Israel and France.
[20] C. Tuncay, ⟨www.arXiv.org/physics/0612137⟩.
[21] S. Redner, A Guide to the First-Passage Processes, University Press, Cambridge, 2001.